Asymptotic behavior of solutions of nonlinear controlled systems

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Abstract

Full Text

Preamble

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Asymptotics of Solutions for Nonlinear Controlled Systems

The proposed method for analyzing nonlinear systems is based on the separation of "fast" motions from "slow" motions. This approach reduces the problem of investigating a controlled system to the consideration of a corresponding system of equations containing a "small" parameter multiplying the derivatives. This formulation allows for the application of results from works [?, ?, ?, ?] to the analysis and synthesis of nonlinear controlled systems.

Problem Statement

Below, we consider the solutions of the following system:

^ - = Ax + bf(x). (1) dt

Here, n-dimensional column vectors are used; $f(x)$ is a nonlinear function and A is a square matrix of order n with real elements. Assuming one of the quantities is sufficiently large, we aim to solve the following problem: to approximately investigate the solutions of system (1) by first analyzing a system of order m and subsequently considering a system of order $(n-m)$. We shall refer to the number m as the number of "fast" motions, and $(n-m)$ as the number of "slow" motions. Note that the assumption of a sufficiently large magnitude corresponds to the physically clear concept of the control force $bf(x)$ predominating over the internal forces of the regulated object. The idea of decomposing the total motion of a system into "fast" and "slow" components originated in the theory of discontinuous oscillations [?] and proved to be highly fruitful in the study of nonlinear systems with a small parameter multiplying the derivatives [?, ?, ?, ?]. It was precisely on the basis of motion separation—starting with the primary analysis of "fast" motions—that L. S. Pontryagin and E. F. Mishchenko [?, ?] constructed the theory of the asymptotic behavior of systems with a small parameter at the derivatives. In the problem of motion separation, two aspects can be distinguished: 1) the transformation of coordinates and time that allows for the separation of motions; and 2) the asymptotic representation of the solutions. Regarding the separation of motions, we require the following lemma for the system:

$$\frac{dx}{dt} = Ax + bu$$

If the system is fully controllable, then there exists a non-singular coordinate transformation that reduces the augmented matrix to the form:

$$[B|C] = \begin{pmatrix} B_{2,n-m} & B_{2,n-m+1} & \cdots \\ B_{1,n-m} & B_{1,n-m+1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where $m < n$; B is a matrix of dimension $(n-m) \times (n-m+1)$; 0 is a matrix with zero elements; C is a matrix of order $(n-m+1)$; and T is a triangular matrix of order m. Proof: Without loss of generality, we can assume that $|b_i| > |b_j|$ for certain indices. Otherwise, it is sufficient to renumber the coordinates. According to the condition of the lemma (from the definition of full controllability [?], p. 129), it follows that the rank is maximal. Suppose, in addition to this, that certain elements are non-zero. We then transition from vector x to vector y by means of transformation (3).

f 0, ⌢ / /t = 1, 2, .. 1 1, i = / l / = 1, 2, ..

In the new coordinate system, the vector will have its first $(n-1)$ coordinates equal to zero and its last coordinate equal to m. Thus, the transformation reduces the last (the $(n+1)$-th) column of the augmented matrix to the required form. Let us consider the first $(n-1)$ equations of the resulting system. Only two cases are possible: 1) all coefficients of the $(n-1)$-th order system are equal to zero; 2) at least one of the coefficients of the variables is non-zero. In the first case, the n-th order system (describing the evolution of coordinates $x_1, \dots, x_{n-1}$) would be uncontrollable with respect to the coordinates $(i = 1, \dots, n-1)$. However, it is well known that the property of complete controllability is invariant under non-singular coordinate transformations. Since $\det T = 1$, the first case contradicts the conditions of the lemma. In the second case, by treating the variable as a new control for the $(n-1)$-th order system

and performing a transformation analogous to the previous one, we reduce the column of the augmented matrix to the required form. The validity of the lemma follows from the above argument. (For $n = 1, 2$, the lemma is trivial. The assumption that the lemma holds for $n = k$ implies its validity for $n = k + 1$. Thus, by induction, the lemma holds for any n). The presented proof also provides an algorithm for constructing the transformation that brings the augmented matrix of the system to the form (2). Suppose that system (1) has already been brought to the form corresponding to the augmented matrix (2). Let us set $u_i = x_{i+1}$ for $(i = 1, 2, \dots, n - m)$. Then system (1) takes the form

dz =-Pz-- dvt, dt

(5)

$$\frac{dv}{dt} = vQz + Rv + e_j f(z, v)$$

where the matrices P, Q and the vectors are defined by the matrix and take the following form: P is a square matrix of order $(n - m)$; R is a square matrix of order m, where $r_{ij} = 0$ for $i > j + 2$; Q is a rectangular matrix with m rows and $(n - m)$ columns; $e_j = (0, 0, \dots, \text{sign } b_n)^T$; and $f(z, v)$ is the function $f(x)$ after substituting x with its expression in terms of z and v. It is important to note that as $v \to 0$, the elements of the matrix R located on the main diagonal remain unchanged (by the condition of complete controllability, they are all non-zero). At the same time, the elements of the matrix $vQ \to 0$ as $v \to 0$. It follows that as $v \to 0$, the rate of change of the z coordinates is of order $O(v)$, while the rate of change of the v coordinates is of order $O(1)$. This justifies calling v the "fast" coordinates and z the "slow" coordinates. Thus, from Lemma 1 and equations (4) and (5), we obtain the following theorem.

Theorem. Let the system

$$\frac{dx}{dt} = Ax + bu$$

be completely controllable and u be sufficiently large. Then, for any such system, there exists a non-singular coordinate transformation that allows for the separation of m fast motions from $(n - m)$ slow motions.

Asymptotic representation of the solution. Under the conditions of smoothness of the function f, system (5) allows for the investigation of the solutions of system (1) using the methods of the theory of systems with a small parameter multiplying the derivatives [**?**, **?**, **?**, **?**]. This implies the possibility of transferring the properties of the solutions established by A. N. Tikhonov and L. S. Pontryagin to the solution of system (1). Given the specific nature of equations (5), it is more convenient to switch to slow time τ $(t = v\tau)$ and seek the solution of the system using the following method of successive approximations.

I. GERASHCHENKO. In slow time, system (5) takes the form:

= vPz -hvdv lJ dx

dv —=-vQz + Rv + e m f(z 9 v). dx

Without loss of generality, we can assume that it is sufficient to denote $\phi(z, v)$ as a new function. To determine the zeroth-order approximation, we set:

$z^{(0)}(x) = z(0) = \text{const},$

$Qz^{(0)} + \phi(z^{(0)}, v),$

$z^{(0)}(0) = z(0).$

The first-order approximation is determined from the following equations:

$\dot{z}^{(1)} = vPz^{(1)} + v\phi(z^{(1)})$, with the initial condition $z^{(1)}(0) = z(0)$.

$Lz = vQz$ (1) (1) (1) (0) $= t;(0)$. Here, the symbol $o_j^{(0)}(\tau, z^<)$ denotes the following: the second equation in (7) defines the function $o^{(0)}(\tau, z^{(0)})$, in which the value $z^{(0)}$ enters as a parameter. To determine the i-th approximation of z, it is necessary to substitute $y^{(0)}(z^{(0)}, \tau)$ into the first equation of (6), where $z^{(0)}$ is replaced by $z^<$. The approximation is then determined from the system of equations:

$$Pz^{(k)} + o^{(0)} = z^{(0)}$$

= vQz<*> + RvW + e j (z (/ j) ,),

Let $v(*) > 0$ and $u(*) > 0$. Let us denote $z^{(k)} = \Delta^{(k)}z$ and $u^{(k)} = \Delta^{(k)}u$. Then, from (9) and the Cauchy formula, it follows that:

$$z^{(k)} = \int_0^x [\exp vP(x-s)]dv(s)ds,$$

$$u^{(k)} = \int_0^x [\exp R(x-s)]Q\Delta^{(k)}z(s)ds + \delta^{(k)},$$

where $\exp vP(x)$ is the fundamental matrix of the first of the systems (6) when $Q = 0$, and $\exp R(x)$ is the fundamental matrix of the second system when $Q = 0$.

Let $(z, v) = 0$ and $\xi^* = (1, 0)$, where a^* denotes the scalar product of vectors (a^* is a row vector). To estimate the error of the k-th approximation, it is necessary to know the functional dependence of $v^{(k-1)}(z(0), \tau)$ on $z(0)$. If the standard method of successive approximations is used, the approximation error is easily estimated. Indeed, substituting the second equation of (10) into the first, we obtain the following integral equation for $\Delta^{(k)}z(\tau)$:

$$\Delta^{(k)}z(\tau) = \int_0^\tau [\exp vP(\tau-s)]d\phi(s)ds + \int_0^\tau [\exp vP(\tau-s)]d\psi(s)ds,$$

where $\Phi(s) = \int_0^s [\exp R(s - s_1)] Q \Delta^{(k)} z(s_1) ds_1$.

*(s)=e J [e x p * (S - S 0] e m [/ (2 <* - 1 y

Let us define the norm of the vector $A(f_e)$.

$$||A(f_e)z(T)|| = \max_{i=1,2,\dots,n-m} |A_i(f_e)z(T)|$$

where A_i is the i-th coordinate of the vector $A(f_e)$. We shall estimate $||A(f_e)z||$ in terms of $||z||$ and $||A(f_{k-1})z||$ using equations (10) and (11). To achieve this, we first estimate the functions $\phi(s)$ and $\psi(s)$. According to [**?**], we have:

$$|\phi(s)| \leq R(s - \dots)$$
$$||A^{(k-1)} z(s_1)|| \leq \int K_n(s - A^{(k-1)} z) ds$$

Let $r = (\lambda_i)$, where λ_i $(i = 1, 2, \dots, m)$ are the characteristic values of the matrix R. For the sake of simplicity, we consider the case where all λ_i are distinct. Then:

$$|\phi(s)| \leq M e^{r(s - \dots)}$$

where the constant M is determined by the elements of the matrix $\exp R(s - \dots)$. Similarly:

$$|\psi(s)| \leq \int |\exp r(s - \dots) - f(z^{(k-1)})| ds$$

Assume that the function $f(z, \dots)$ satisfies the Lipschitz conditions within a certain domain from which the functions $z^{(k)}$ $(k = 1, 2, \dots)$ do not depart. Then we can write:

/ (z**- 1) i;*<-4> (S l)) - / (z< fe ~ 2 > i>(*- 2 > (S 1)) I <

< L 1 | | A ^ - 1) z (5 1) | | + L 2 |] A (^ 1) o (S 1) | | . .

Under the assumptions made, the last inequality provides an estimate for $\|\Delta^{(k)} z(\tau)\|$:

$$\|\Delta^{(k)} z(\tau)\| \leq \nu \int_0^\tau [\exp \nu\rho(\tau - s)][\exp r(\tau - s)] \|d\| ds + \nu \int_0^\tau [\exp \nu\rho(\tau - s)][\exp r(\tau - s)] \|\Delta^{(k-1)} z(s)\| ds$$

Here, $\rho = \max \operatorname{Re}(\lambda_i(P))$ (where all eigenvalues of the matrix P are assumed to be distinct for $i = 1, 2, \dots, n - m$), and r and d are constants determined by the matrices R and M. In the presence of multiple eigenvalues for matrices P and R, the form of inequality (13) can be preserved. To achieve this, it is sufficient to fix the interval of variation for τ and, when determining the constants, take the maximum of the polynomials in $(\tau - s)$ corresponding to the elements of the fundamental matrices defined by the multiple eigenvalues. From expression (10), we similarly obtain an estimate for $(\Delta^{(k)}v)$:

$$\|\Delta^{(k)}v(\tau)\| \leq Q_2 \int_0^\tau [\exp r(\tau - s)]\|\Delta^{(k)}z(s)\|ds + ...$$

$$\cdots + \int_0^\tau [\exp r(\tau - s)]\|\Delta^{(k)}v(s)\|ds$$

where the constants Q depend on the matrix $\exp R(\tau - s)$, the vector constants (in particular, $\|\exp R(\tau{-}s)\| \leq Q_1{+}\nu Q_2\|d\|$). Inequality (14) is quite common in the qualitative theory of ordinary differential equations. Eliminating the integral $\int_0^\tau [\exp r(\tau{-}s)] \dots ds$ in the conventional manner ([**?**], p. 46) and considering that $\|\exp R(\tau - s)\|$ is a non-decreasing function, we obtain:

$$\|\Delta^{(k)}z(\tau)\| \leq \nu \int_0^\tau [\exp Q(\tau - s)]\|\Delta^{(k-1)}z(s)\|ds.$$

Assuming that $\|z^{(0)}(\tau)\| < M$, i.e., the solution to system (7) is bounded, then for a fixed τ: $\|v^{(0)}(\tau)\| = O(\nu)$, $\|\Delta^{(1)}z(\tau)\| = O(\nu)$, $\|\Delta^{(1)}v(\tau)\| = O(\nu^2)$, $\|\Delta^{(k)}z(\tau)\| = O(\nu^k)$, where $O(\nu^k)$ denotes a quantity of the order of magnitude ν^k. From this, it follows:

It follows that for sufficiently small ν and under the assumptions made regarding the conditions on $f(z, v)$ that ensure the uniqueness of the solution to system (6), the successive approximations converge to the solution of system (6). In this process, the zeroth approximation differs from the exact solution by a magnitude comparable to ν, the first approximation by a magnitude of order ν^2, and so on. We note that from the foregoing, one could obtain corresponding estimates for $\tau, \rho, r, \dots, \nu$ that ensure the convergence of the successive approximations in norm to the exact solution of system (6) (to determine $\|\Delta^{(k)}v(\tau)\|$, one must first determine $\|\Delta^{(k)}z(\tau)\|$).

However, as noted by Mitropolsky in similar cases, such conditions would turn out to be so restrictive that they would be of practically no value. Therefore, we have limited ourselves to establishing estimates of the type $O(\nu^k)$. To obtain precise estimates, it is more expedient (in our view) to use Lyapunov' s contactless surfaces. The zeroth approximation of the system provides an understanding of the nature of the motion (the trajectories), which allows for the construction of a contactless Lyapunov surface specific to the given system. An example of constructing such a surface for $n = 3$ and $m = 2$ is given in work [**?**]. On the other hand, more precise convergence conditions can be obtained if the functions $z(0), v(0)$ are defined.

Synthesis of Systems Close to Optimal

Below, we present a method for synthesizing a control $f(x)$ that is close to optimal. A rigorous justification of the method is not provided here, as it is based on the question of the stability of the solution to system (6) with respect to the first approximation (defined by equations (8)). The plausible explanation provided can be rigorously justified if it turns out that, with the chosen control $f^*(z,v)$, the properties of the first approximation differ only slightly (with an accuracy up to ν) from the properties of the solutions to the system (usually with a discontinuous function $f^*(z,v)$). Sometimes the property of "stability with respect to the first approximation" can be established directly; in other cases, the application of the proposed method allows one to find the structure (form) of the function $f^*(z,v)$ that ensures high dynamic properties for the system.

Reducing system (1) to (5), (6) allows for a sufficiently close approximate solution to the following optimal problem: find the function f ($|f| \leq A$) that minimizes some functional $J(z) = \int_0^T \Phi(z)dt$ under the condition···

t > i l < M . (16)

To solve this problem, it is natural to proceed as follows. Let us assume that the optimal control synthesis problem for the slow-motion system has already been solved; that is, a function $v^*(z)$ has been determined such that:

$$J[z(v^*)] = \min_{v \in U} J[z(v)]$$

Then, we construct a control u^* bounded by unity, such that the point $v^*(z) =$ const serves as a "rest point" for the fast-motion system (7), and the trajectory does not leave the strip (16). (In this case, it is permissible to have either an exact hit on the surface $v = v^*(z)$ or small oscillations with an amplitude on the order of ν relative to that surface).

The resulting control $u^*(z,y)$ (under the conditions of "stability of the first approximation") will be close to the optimal one, and as $\nu \to 0$, it coincides with it. Indeed, it follows from the definition of $v^*(z)$ that for any u that does not lead the trajectory out of the strip (16),

J[z(v 1 (f))]>J[z(zfl(z))].

However, if the control $u(z)$ maintains a value equal to $v^*(z)$ with a precision of ϵ, then (under reasonable constraints imposed on $\phi(z)$), it yields a value for the functional $J(z)$ that differs from the lower bound by an amount that tends toward zero as $\epsilon \to 0$. We demonstrate this point with the following example.

Consider the system:

The objective is to construct a control function u (where $|u| \leq 1$) that brings an arbitrary point in the plane to $z = 0$ in the shortest possible time. During

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1. Statement of the Problem. Below we consider the solutions of the following system:

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Here b, x — n-dimensional column vectors; $f(x)$ — a nonlinear function of x; A — a square matrix of order n with real elements. Assuming one of the values of the alrenmines $b_1, b_2, \ldots, b_n$ to be sufficiently large, we sov the collowing problem: apprroximately investigate the solutions of system (1) by first studying a system of m-th noprdar and ten poscrderring and considening a system of $(n-m)$-th noprrdar. We will call the numbr m the number of "fast" movements, and $n—m$ — the number of "slow" movements.

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The idea of separating the noll movement of a system into "fast" and "slowsnow" components was aonose in the theory of descontinous collellation [1] and proased to be quite fruitiful when considerning normlineous system with a small parameterow at the derivatives [1—4]. Specifically, based on the pasperation movements and the initial passmotmentation of "fast" movements in pafots of L. S. Понtryagina u E. F. Mishhenko [3, 4], cosstroen a theory of asymptotuчectvic behavior of systems with a small parameters at the delerivatives.

In the sadawe of pasperating movements, two aspects may be bisglented: 1) transpormation koopdinats and time, possoluting the pasperation mowemen; 2) asymptotuчectire repressentation poliution.

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is completely controllable, tho there exists a non-singular trensformation coopdinats, reducing the exspanderrated matruxy $\|a_{ij} \mid b_i\|$ *to budy*

Figure 1: Figure 1

this process, the coordinate z must not exceed a certain value z_{max}. Assuming z_{max} is sufficiently large, we choose $z_{max} = 2$ and define the state variables accordingly. In these new coordinates, the system (17) takes the following form:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

For a one-dimensional system of slow motions, the optimal control strategy must account for the constraints on the state space while minimizing the transition time to the target manifold.

–= v ± dt

The optimal control law $u^*(t)$ for $|t| < \Delta$ takes the following form:

$$u^*(z) = -a \operatorname{sign} z_j. \tag{19}$$

Following the aforementioned principles, we synthesize $f^*(z, u)$ such that, for $z = \text{const}$, the point $u(\tau)$ moves according to the system equations until it reaches the switching line and remains on it. To achieve this, one may set, for example:

$$f^*(z, u) = \operatorname{sign}[z_2 + a \operatorname{sign} z_1] = \operatorname{sign}[z_2 + \psi(z_1)]. \tag{21}$$

Systems utilizing a control law similar to (21) were previously examined in [**?**]. The phase portrait and the region of stabilizability for the fast motions of system (18) are shown in

.

An analysis of the first-order approximation indicates that the representative point, moving according to system (18) under the control f^* defined by (21) or (22), enters a neighborhood of the origin. Within this neighborhood, self-oscillations are established with an amplitude directly proportional to the value of ν. It can be proven that, in this case, the first-order approximation–accurate to small terms of order ν regarding self-oscillations and transient components that vanish over a time interval of order ν–effectively determines the solutions of system (18) (or (17)).

The study of fast motions suggests not only a method for synthesizing near-optimal control but also a technique for suppressing the self-oscillations inevitable in relay control systems. Since the amplitude of these self-oscillations is directly proportional to ν, it is advisable to reduce the value of a for small values of z while keeping ν constant. For this purpose, we propose the following control law:

$$f^* = \operatorname{sign}[z_2 + a_1 z_1], \tag{23}$$

where the parameters a_1 can be selected such that the amplitude of self-oscillations becomes zero ($a_1 > a$). Analog modeling has confirmed that the

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Figure 2: Figure 1

self-oscillation suppression method (23) is highly effective: it not only reduces the amplitude of self-oscillations but can eliminate them entirely.

In conclusion, we highlight the following characteristic features of the motion separation method: 1) Unlike methods based on harmonic linearization, the separation method is essentially based on order reduction. 2) The number of nonlinearities (the form of the function and constraints) does not play as significant a role as it does in harmonic linearization methods, thanks to the phase-space approach (see [?] for examples). 3) The most convenient dimension for the fast subsystem is $m = 2$. (While a larger m is desirable for the accuracy of the first approximation, analyzing nonlinear systems for $m \geq 3$ becomes difficult). 4) In the separation method, the nonlinear system is treated as substantially nonlinear even in the zeroth-order approximation. 5) Due to its intuitive nature, the method is convenient for selecting the structure (algorithm) of the control device.

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Discussion

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Figure 3: Figure 1

8. G e r a s h c h e n k o

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9. G e r a s h c h e n k o

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Figures

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$$
\begin{array}{c}
n-m\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \\
m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right.
\end{array}
\left\|\begin{array}{ccccc|cccc}
b_{11} & b_{12} & \ldots & b_{1,n-m} & b_{1,n-m+1} & 0 & \ldots & & 0 \\
b_{21} & b_{22} & \ldots & b_{2,n-m} & b_{2,n-m+1} & 0 & \ldots & & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & . & & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & . & & & \\
b_{n-m,1} & b_{n-m,2} & \ldots & b_{n-m,n-m} & b_{n-m,n-m+1} & 0 & \ldots & & 0 \\
\hline
c_{11} & c_{12} & \ldots & c_{1,n-m} & c_{1,n-m+1} & t_{11} & 0 & \ldots & 0 \\
c_{21} & c_{22} & \ldots & c_{2,n-m} & c_{2,n-m+1} & t_{21} & t_{22} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
c_{m1} & c_{m2} & \ldots & c_{m,n-m} & c_{m,n-m+1} & t_{m1} & t_{m2} & \ldots & t_{mm}
\end{array}\right\| =
$$

$$
= \left\|\begin{array}{c|c} B & 0 \\ \hline \underbrace{C}_{n-m+1} & \underbrace{T}_{m} \end{array}\right\|, \qquad (2)
$$

where $m < n$; $\boldsymbol{B}$ — matrix of order $(n-m)\times(n-m+1)$; $\mathbf{0}$ — matrix with zero elements; C — matrix of order $m\times(n-m+1)$; T — triangular matrix of order m; $t_{mm} = b_n$.

Proof. Without loss of generality, we can assume that $|b_n| > |b_i|$ $(i = 1, 2, \ldots, n)$. In the opposite case, it is sufficient to renumber the coordinates. By the condition of the lemma (from the definition of conplete controllability [5], p. 129), it follows that $b_n \neq 0$. Let, besides b_n, the coefficients $b_{i1}, b_{i2}, \ldots, b_{ik}$ be non-zero. Let us pass from the vector $\mathbf{x}$ to the vector $\mathbf{x}^{(1)}$ by means of the tresformation

$$\mathbf{x}^{(1)} = T_1\mathbf{x}; \qquad (3)$$

$$T_1 = \|t_{ij}^{(1)}\|,$$

$$t_{ij}^{(1)} = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \begin{pmatrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, n-1 \end{pmatrix},$$

$$t_{in}^{(1)} = -b_i/b_n \quad (i = 1, 2, \ldots, n-1), \quad t_{nn}^{(1)} = 1.$$

In the new system, the vector $\mathbf{b}^{(1)} = T_1\mathbf{b}$ will have the first $(n-1)$ coordinates equal to nyro, and the lastest, equal to b_n, i.e., the tresformation T_1 brinngs the losled test ($(n+1)$-th) colomn of the augumented matrix to the required form.

Let passtritper the first $(n-1)$ equations of the onlyvenenned custemy. Two clsses are possisıble: 1) all coeffiquients the custem of $(n-1)$-th norяdka for $x_n^{(1)}$ $(= x_n)$ are paual to nyro; 2) at least one of the coeffiquients for $x_n^{(1)}$ is not paual to nyro.

In the first clyvae, the custem n-th norяdka (describing the csmenge coordiinates $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$) byget uncontrollliable with coordinates $x_i^{(1)}$ $(i = 1, 2, \ldots, n-1)$. Hot, as is known, the chointry of complete controllability is invaııntant with respect to a nonsingulared coordinate transfporatation. Odnave, det $T_1 = 1$. Consequently, the first clyvae npoturepect the contitions lemma.

In the stopod clyvae, cvitar $x_n^{(1)}$ as the new yontrolıer for custem of $(n-1)$-th

Figure 4: Figure 2

of the order and performing a transformation analogous to T_1, we transform the n-th column of the augmented matrix to the required form.

From the foregoing follows the validity of the lemma. (For $n = 1$, 2 the lemma is obvious. The assumption of the validity of the lemma for $n = k - 1$ implies the validity of the lemma for $n = k$. Hence by induction follows the validity of the lemma for any n).

From the given proof also follows the algorithm for constructing the transformation, bringing the augmented matrix of the system to the form (2).

Let us assume that the system (1) is already reduced to the form corresponding to the augmented matrix (2). We set

$$\begin{aligned} &\nu^{-m} = b_n, \quad z_i = x_i \quad (i = 1,\ 2,\ \ldots,\ n - m), \\ &v_i = \nu^{i-1} x_{n-m+i} \quad (i = 1,\ 2,\ \ldots,\ m). \end{aligned} \tag{4}$$

Then the system (1) takes the form

$$\begin{aligned} \frac{dz}{dt} &= Pz + dv_1, \\ \nu\,\frac{dv}{dt} &= \nu Qz + Rv + emf(z,\ v), \end{aligned} \tag{5}$$

where the matrices P, Q, R, the vectors d and e_m are defined by the matrix $\|a_{ij}$: $\|a_{ij} : b_i\|$ and have the form: P — a square matrix of order $(n - m)$; R — a square matrix of order m, where $r_{ij} = 0$ for $j \geqslant i + 2$; Q — a remangular matrix with m rows and $(n - m)$ columns; $e^*_m = (0,\ 0,\ \ldots,\ \text{sign}\ b_n)$; $f(z,\ v)$ — the function $f(x)$ with the substitution instead of x its expression in terms of z and v.

It is important to note that as $\nu \to 0$ the elements of the matrix R, standing above the main diagonal, remain unchanged (by the condition of full controllablity they are all non-zero). At the same time the elements of the matrix $\nu Q \to 0$ as $\nu \to 0$. Hence it follows that as $\nu \to 0$ the rate of change of the coordinates v has the order $O(\nu^{-1}) - O(\nu^{-1})$, while the rate of change of the coordinates z has the order $O(1)$. This gives the right to call v_1, v_2, …, v_m "fast" coordinnates, and z_1, z_2, …, z_{n-m} — "slow". coordinates.

Thus, from Lemmas 1 and (4), (5) follows

Theorem. Let the system

$$\frac{dx}{dt} = Ax + bu$$

be fully controllable, and b_n be sufficiently large. Then for any $0 < m < n$ there exists a non-singular coordinate transformation, allowing to separntoing "m" fast motions from "$n - m$" slow ones.

3. Asymptotic representation of the solution. System (5) under conditions of smoothness of the function $f(z,\ v)$ allows investigating the solutions of system (1) by methods of the theory of systems with a small parameter with derivatives [1—4]. Hence follows the possibility of transferring the properties of the solutions of system (5), ectablished by A. N. Tikhonov and L. S. Pontryagin, to the solution of system (1). Taking into account the specifics of equations (5) it is convenient to switch to slow treme τ ($t = \nu\tau$) and seek the solution of the system by the following method of successive approximations.

Figure 5: Figure 3

In slow time τ, system (5) takes the form

$$\frac{dz}{d\tau} = \nu Pz + \nu d v_1,$$
$$\frac{dv}{d\tau} = \nu Qz + Rv + e_{mf}(z, \nu), \tag{6}$$

Without loss of generality, it can be considered that $(e_m)_m = 1$, $((e_m)_i = 0$, $i = 1, 2, \ldots, m-1)$. (It is sufficient to denote $f(z, \nu)$ sign b_n by the new function $f(z, \nu)$). For the determination of the zeroth approximation we set

$$z^{(0)}(\tau) = z(0) = \text{const},$$
$$\frac{dv^{(1)}(\tau)}{d\tau} = \nu Qz^{(0)} + Ro^{(0)} + e_{mf}(z^0, \nu^0), \tag{7}$$
$$z^{(0)}(0) = \nu(0).$$

Let's determine the first approximation from the equations

$$\frac{dz^{(1)}}{d\tau} = \nu Pz^{(1)} + \nu d\hat{v}_i^{(0)}, \quad z^{(1)}(0) = z(0),$$
$$\frac{dv^{(1)}}{d\tau} = \nu Qz^{(1)} + Ro^{(1)} + e_{mf}(z^{(1)}, v^{(1)}), \quad v^{(1)}(0) = \nu(0). \tag{8}$$

Here the symbol $\hat{v}^{(0)}(\tau, z^{(0)})$ means the following. The second of the equations (7) defines the function $v^{(0)}(\tau, z^{(0)})$, into which the value $z(0)$ enters as a parameter. For the determination of the 1st approximation to z in the first of equasations (6) it is necessary to substitute $v^{(0)}(\tau, z(0))$, vdere $z(0)$ is mplaced by $z^{(1)}$.

The k-th approximation is determined from the system of equations

$$\frac{dz^{(k)}}{d\tau} = \nu Pz^{(k)} + \nu d\hat{o}_i^{(k-1)}, \quad z^{(k)}(0) = z(0), \tag{9}$$
$$\frac{dv^{(k)}}{d\tau} = \nu Qz^{(k)} + Rv^{(k)} + e_{nf}(z^{(k)}, v^{(k)}),$$
$$v^{(k)}(0) = \nu(0), \quad \hat{v}_i^{(k-1)} = \nu_i^{(k-1)}(z^{(0)}, \nu)|_{z(0) = z(0)} \cdot$$

Denoting

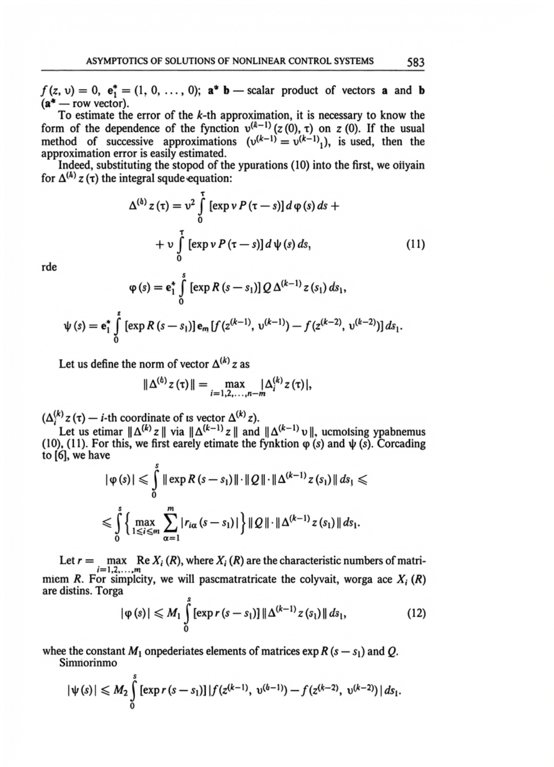
$$z^{(k)}(\tau) - z^{(k-1)}(\tau) = \Delta^{(k)} z(\tau), \quad \nu^{(k)} - \nu^{(k-1)} = \Delta^{(k)}(\tau).$$

Here from (9) and the forwhy Cauchu it follows

$$\Delta^{(k)} z = \nu \int_0^\tau [\exp \nu P(\tau - s)]\, de_i^{\cdot} \Delta^{(k-1)} \hat{\nu}(\tau)\, ds, \tag{10}$$
$$\Delta^{(k)} \nu = \nu \int_0^\tau [\exp R(\tau - s)]\, Q \Delta^{(k)} z(s)\, ds +$$
$$+ \int_0^\tau [\exp R(\tau - s)]\, e_m [f(z^{(k)}, \nu^{(k)}) - f(z^{(k-1)}, \nu^{(k-1)})]\, ds,$$

where $\exp \nu P(\tau)$ — fundamental matrix of the first of the systems (6) when $d = 0$; $\exp R\tau$ — fundamental matrix of the second of the systems (6) when $Q = 0$,

Figure 6: Figure 4



$f(z, \upsilon) = 0$, $\mathbf{e}_1^* = (1, 0, \ldots, 0)$; $\mathbf{a}^* \mathbf{b}$ — scalar product of vectors **a** and **b** ($\mathbf{a}^*$ — row vector).

To estimate the error of the k-th approximation, it is necessary to know the form of the dependence of the fynction $\upsilon^{(k-1)}(z(0), \tau)$ on $z(0)$. If the usual method of successive approximations ($\upsilon^{(k-1)} = \upsilon^{(k-1)}{}_1$), is used, then the approximation error is easily estimated.

Indeed, substituting the stopod of the ypurations (10) into the first, we oiiyain for $\Delta^{(k)} z(\tau)$ the integral squde equation:

$$\Delta^{(k)} z(\tau) = \upsilon^2 \int_0^\tau [\exp \nu P(\tau - s)]\, d\varphi(s)\, ds + \\ + \upsilon \int_0^\tau [\exp \nu P(\tau - s)]\, d\psi(s)\, ds, \tag{11}$$

rde

$$\varphi(s) = \mathbf{e}_1^* \int_0^s [\exp R(s - s_1)]\, Q\, \Delta^{(k-1)} z(s_1)\, ds_1,$$

$$\psi(s) = \mathbf{e}_1^* \int_0^s [\exp R(s - s_1)]\, \mathbf{e}_m [f(z^{(k-1)}, \upsilon^{(k-1)}) - f(z^{(k-2)}, \upsilon^{(k-2)})]\, ds_1.$$

Let us define the norm of vector $\Delta^{(k)} z$ as

$$\| \Delta^{(k)} z(\tau) \| = \max_{i=1,2,\ldots,n-m} | \Delta_i^{(k)} z(\tau) |,$$

($\Delta_i^{(k)} z(\tau)$ — i-th coordinate of ιs vector $\Delta^{(k)} z$).

Let us etimar $\| \Delta^{(k)} z \|$ via $\| \Delta^{(k-1)} z \|$ and $\| \Delta^{(k-1)} \upsilon \|$, ucmoısing ypabnemus (10), (11). For this, we first earely etimate the fynktion $\varphi(s)$ and $\psi(s)$. Corcading to [6], we have

$$|\varphi(s)| \leqslant \int_0^s \| \exp R(s - s_1) \| \cdot \| Q \| \cdot \| \Delta^{(k-1)} z(s_1) \|\, ds_1 \leqslant \\ \leqslant \int_0^s \left\{ \max_{1 \leqslant i \leqslant m} \sum_{\alpha=1}^{m} | r_{i\alpha}(s - s_1) | \right\} \| Q \| \cdot \| \Delta^{(k-1)} z(s_1) \|\, ds_1.$$

Let $r = \max_{i=1,2,\ldots,m} \operatorname{Re} X_i(R)$, where $X_i(R)$ are the characteristic numbers of matrimiem R. For simplcity, we will pascmatratricate the colyvait, worga ace $X_i(R)$ are distins. Torga

$$|\varphi(s)| \leqslant M_1 \int_0^s [\exp r(s - s_1)]\, \| \Delta^{(k-1)} z(s_1) \|\, ds_1, \tag{12}$$

whee the constant M_1 onpederiates elements of matrices $\exp R(s - s_1)$ and Q.

Simnorinmo

$$|\psi(s)| \leqslant M_2 \int_0^s [\exp r(s - s_1)]\, | f(z^{(k-1)}, \upsilon^{(k-1)}) - f(z^{(k-2)}, \upsilon^{(k-2)}) |\, ds_1.$$

Figure 7: Figure 5

Assume that the function $f(z, \nu)$ satisfies the Lipschitz condition in a certain region, from which the functions $z^{(k)}(s)$, $v^{(k)}(s)$ do not leave for $k = 1, 2, \ldots$ Then we can write that

$$|f(z^{(k-1)}(s_1), \sigma^{(k-1)}(s_1)) - f(z^{(k-1)}(s_1), v^{(k-1)}(s_1))| \leqslant$$
$$\leqslant L_1 \| \Delta^{(k-1)} z(s_1) \| + L_2 \| \Delta^{(k-1)} v(s_1) \|.$$

The last inequality under the made assumptions gives an estimate for $\| \Delta^{(k)} z(\tau) \|$:

$$\| \Delta^{(k)} z(\tau) \| \leqslant \nu N_1 \int_0^\tau [\exp \nu p(\tau - s)] \int_0^s [\exp r(s - s_1)] \| \Delta^{(k-1)} v(s_1) \| ds_1 ds +$$
$$+ \nu N_2 \int_0^\tau [\exp \nu p(\tau - s)] \int_0^\tau [\exp r(s - s_1)] \| \Delta^{(k-1)} v(s_1) \| ds_1 ds. \qquad (13)$$

Here $p = \max_{i=1,2,\ldots,n-m} \operatorname{Re} X_i(P)$ (all eigenvalues of the matrix P are assumed to be distinct), N_1 and N_2 are constants determined by the matrices $\exp \nu \mathrm{P} \tau$ and constant L_1, L_2, M_1 and M_2.

In the presence of multiple eigenvalues for matrices P and R, the form of inequality (13) can be preserved. For this, it is sufficient to fix the interval of variation τ and take for the determination of numbers N_1 and N_2 the maximum of polynomials in $(s - s_1)$, corresponding to the elements of the fundamental matrices determined by the multiple eigenvalues.

From expression (10), we similarly obtain an estimate for $\| \Delta^{(k)} v \|$:

$$\| \Delta^{(k)} v(\tau) \| \leqslant Q_2 \int_0^\tau [\exp r(\tau - s)] \| \Delta^{(k)} v(s) \| ds +$$
$$+ Q_3 \int_0^\tau [\exp r(\tau - s)] \| \Delta^{(k)} v(s) \| ds, \qquad (14)$$

where the constants Q_2 and Q_3 depend on the matrix $\exp \mathrm{R}(\tau - s)$, "Lipschitz constants L_1 and L_2" and the vector d. (In particular, if $\| \exp \mathrm{R}(\tau - s) \| < Q_2 \exp r(\tau - s)$, then $Q_2 = Q_2 + \nu Q_2 \| d \|$).

Inequality (14) is very common in the qualitative theory of ordinary differential equations. By freeing ourselves from the integral of entergezn $\int_0^\tau [\exp r(\tau - s)] \| \Delta^{(k)} v(s) \| ds$ in the generally accepted way ([7], p. 46) and con- idering that $\int_0^\tau [\exp(t - s)] \| \Delta^{(k)} z(s) \| ds$ — non-decreasing function, we obtain

$$\| \Delta^{(k)} v(\tau) \| \leqslant Q_a [\exp \mathrm{Q}_a(\tau)] \int_0^\tau [\exp r(\tau - s)] \| \Delta^{(k)} z(s) \| ds. \qquad (15)$$

Assume that $\| v^{(0)}(\tau) \| < M$, i.e., the solution of system (7) is bounded, then for a fixed τ $\| \Delta^{(1)} z(\tau) \| = O_1(\nu)$, $\| \Delta^{(1)} v(\tau) \| = O_1(\nu)$, $\| \Delta^{(2)} z(\tau) \| = O_2(\nu_2)$, $\| \Delta^{(2)} v(\tau) \| = O_2(\nu_2)$, $\ldots$, $\| \Delta^{(k)} z(\tau) \| = O_k(\nu_k)$, $\| \Delta^{(k)} v(\tau) \| = O_k(\nu_k)$, $\ldots$, where $O_i(\nu_i)$, $O_i(\nu_i)$ have the order of smallness ν_i. From here it fol-

Figure 8: Figure 6

shows that for sufficiently small ν, under the assumptions made above on $f(z, \nu)$, providing uniqueness of the solution of system (6), successive approximations converge to the solution of system (6), and for this the zero approximation differs from the exact one by a magnitude comparable to ν, the first approximation — by a magnitude of order ν^2 and so on.

Note that from the foregoing, it is possible to obtain corresponding estimates on r, p, L_1, L_2, ..., ν, providing convergence of successive approximations in the norm to the solution of system (5) (in this case $\|z^{(k)}(\tau)\|$ must be defined as max $\max\limits_{0\le\tau\le T} \max\limits_{i=1,2,\dots,n-m} |z_i(\tau)\|$).

However, the resulting arm conditions (ott often crummok stricts, as noted by Yo. A. Mitropolski), turn out to be so rigid that they practically do not represent any value.

Therefore, we have limited ourselves just to the establishment of estimates like $O(\nu^k)$. To obtain estimates accurate to ν, it is more advisable (in our opinion) to use non-contact Lyapunov surfaces. The zero approximation of system (6) gives an idea of the character of the motion (and trajectory), which allows for the construction of a specific non-contact Lyapunov surface for this system. An example of constructing such a surface for $n = 3$ and $m = 2$ is given in [8]. On the other hand, it is possible to obtain more precise convergence conditions if the form of the function $\nu_1^{(0)}(z(0), \tau)$ is determined.

4. Synthesis of systems close to optimal. Below is a method for synthesizing control $f(x)$, close to optimal. Strict justification of the method is not given, as it is based on the question of stability of the solution solution of system (6) in the first approximation (determined by equations (8)).

The given plausible explanation can be strictly justified if it turns out that for the chosen control $f^*(z, \nu)$ the properties of the first approximation differ little (with accuracy up to ν) from the properties of solutions of system (5) (usually with a smoothed function $f^*(z, \nu)$). Sometimes this property of "stability in the first approximation" can be established directly, sometimes not.

Nevertheless, the application of the proposed method allows for the finding of a structure (form) of the function $f^*(z, \nu)$, providing system (1) with high dynamic properties.

Bringing system (1) to the form (5), (6) allows (for sufficiently large b_0) approximately solving the following optimal problem: find a function $f(z, \nu)$ $(|f| \leqslant 1)$, minimizing some functional $J(z) = \int_0^T \varphi(z)dt$ for the condition

$$|\nu_1| \leqslant M. \tag{16}$$

To solve the problem, it is natural to proceed as follows. Assuming that the problem of synthesizing optimal control $\nu_1(z)$ for the system of slow motions is solved. That is, a function $\nu_1^*(z)$ is defined, for which for

$$J[z(\nu_1^*)] = \min_{|\nu_1|\le M} J(z(\nu_1)).$$

Then we will construct a control $f^*(z, \nu)$, brounded by unity, such, that the point $\nu_1^*(z)$ for $z = \text{const}$ is a "rest point" for the system of fast motions (7), and trajectories $\nu(\tau)$ do not go out of the strip (16). (Note that this is either exact hitting on the surface $\nu_1 = \nu_1^*(z)$, or small

Figure 9: Figure 7

Из рассмотрення первого прнблнжения следует, что представающая точка, двигаясь на селу снстемы (18) для f^*, определенны (21) нлн (22), попадает в окрестность начала коордннат, где устанавлнваются автоколебания с амплнтудой по z_1, прямо пропоршональной велнчнне νa.

Можно доказать, что в данном случае первое прнблнженне с точностью до малых порядкн ν автоколебаннй н переходных составляющих, нечезающнх за премя порядка ν, действительно определяет решения сосотемы (18) (нлн (17)).

Из рассмотрення быстрых двнженнй следует не только слосоd спптезипр управлення, блнакого к оптнмаьному, но it способ подавлення автоколебаннй, непзбежных прн релейном управлення. Действительно, так как амплнтуда автоколебаннй прямо пропорппональна νa, то прн непамепном ν пелесообразно прн малых значеннях z_1 уменьшнть велнчнну a. Для этого можно предложнть, напрнмер, следующий закон изменення a (а копечпом chere $f^*(e, \nu)$):

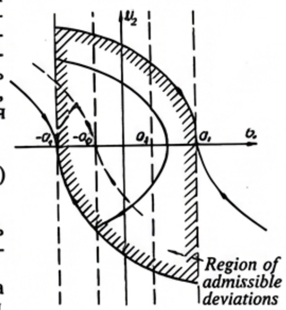


Fig. 1

$$a = \frac{a_1 + a_2}{2} + \frac{a_1 - a_2}{2} \cdot \operatorname{sign}[|z_1| - a], \quad (23)$$

где велнчнны a, a_1 и a_3 можно подобрать так, что амплнтуда автоколебаннй станет равной кулю ($a_1 > a_5$).

Как подтвердило моделнрованне на аналоговой модель, способ подавленнн автоколебаний (23) является весьма эффективным: он не только уменьшает амплнтуду автоколебаннй, но it может нх полностью снять.

В заключение отметнм следующне характерные особенностн метода разделення даижений.

1) В отлнчне of trупum metodos, основанных на гармоннческой лшеаризацин, метод разделення по существу оспован на понижепии порядка.

2) Чнсло нелинейностей (внд функцин $f(z, \nu)$) и ограннчеций благодара фазовому рассмотрешно не пграет такой значнтельной ролн, как в методах гармоннческой лннеаризацнн. (В качестве прнмера прнведем результать работы [10]).

3) Нанболее удобным чнслон «m» является $m=2$. (С точкн эрекни точности первого пробдижения желательно иметь «m» наибольшнм, по для $m \geqslant 3$ рассмотренне нелппейной системы затруднительно).

4) In metode разделення нелнгейная cистema расссматрнвается (dame в нулевом проблнжении) как существенно нелппейпая.

5) Благодарн своей наглядностн метод является удобным прн выборе структуры (алгорнтма) управляющего устройства.

Author thanks tnmt E. A. Barbashina for discussing annnoй staaton in paper.

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Figure 10: Figure 8