ELLIPTIC (CO)BOUNDARY MORPHISMS

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ELLIPTIC (CO)BOUNDARY MORPHISMS

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1. Introduction. The theory of boundary-value problems on closed manifolds whose boundary (in the sense of ([3])) contains submanifolds of different dimensions was studied in ([2], [3]), where criteria necessary and sufficient for the Fredholm property of the corresponding problem were indicated. At the same time, if the finite-dimensionality of the kernel of the operator followed from an a priori estimate, then for the finite-dimensionality of the cokernel it was necessary to construct explicitly an operator almost inverse* to the given one. The asymmetry of such an approach is obvious and is dictated by the circumstance that the adjoint problem does not have the same structure as the original one. Establishing an a priori estimate for the adjoint problem is a much more difficult matter.

In the present paper a new class of operators is introduced—elliptic (co)boundary morphisms. The set of such morphisms is naturally endowed with the structure of a $*$-algebra, so that from the a priori estimate ([5]) all the required properties of the operator follow immediately and, first of all, the finiteness theorem. We emphasize that although formally boundary-value problems (in the sense of ([2], [3])) are not a specialization of elliptic morphisms, nevertheless the finiteness theorem for these morphisms includes, as a special case, the finiteness theorem for boundary-value problems.

2. Boundary and coboundary operators. Let X be a closed C^∞ manifold of dimension N, and let Y be its submanifold of codimension ν. Let $E \to X$ be a complex vector bundle with finite-dimensional fiber. Choose some Riemannian metric on X and denote by $\Gamma^s(X, E)$ the Sobolev space ([1]) of sections of the bundle E (s is an arbitrary real number).

Definition 1. The elementary boundary operator b_Y is the restriction morphism

$$b_Y : \Gamma^s(X, E) \to \Gamma^{s-\nu/2}(Y, E|Y), \qquad s > \nu/2,$$

which assigns to each section $f \in \Gamma^s(X, E)$ its restriction $f|Y$ to the submanifold Y.

Definition 2. The elementary coboundary operator β_Y is the morphism adjoint to the morphism b_Y, i.e. the morphism

$$\beta_Y : \Gamma^{-s+\nu/2}(Y, E|Y) \to \Gamma^{-s}(X, E), \qquad s > \nu/2,$$

defined by the formula

$$(\beta_Y f, g) = (f, b_Y g),$$

where $g \in \Gamma^s(X, E)$.

Remark. If $X = \mathbf{R}^n$, and the manifold Y consists of one point $Y = \{0\}$, then

$$b_Y f = (\delta, f), \qquad \beta_Y \lambda = \lambda\delta.$$

Let $E_i \to X$, $F_i \to Y$, $i = 1, 2$, be complex vector bundles over the manifolds X and Y, respectively, and let

$$D_{XX} : \Gamma^s(X, E_1) \to \Gamma^t(X, E_2),$$

$$D_{YY} : \Gamma^{t+r+\nu/2}(Y, F_1) \to \Gamma^{s-r-\nu/2}(Y, F_2)$$

* For the definition of an almost inverse operator, see Sec. 5.

morphisms defined on the manifolds X and Y, and

$$D_{XY} : \Gamma^s(X, E_1) \to \Gamma^{s-p-\nu/2}(Y, F_2),$$

$$D_{YX} : \Gamma^{t+\dot{r}+\nu/2}(Y, F_1) \to \Gamma^t(X, E_2)$$

are boundary and, respectively, coboundary morphisms. The morphisms D_{XX} and D_{YY} are (pseudo)differential operators, the morphism D_{XY} is a composition of a (pseudo)differential and an elementary boundary operator, and the morphism D_{YX} is a composition of an elementary coboundary and a (pseudo)differential operator.

Proposition 1. *Let π and ρ be the orders of the morphisms D_{XY} and D_{YX}, respectively, in the "transversal" direction. Then, for*

$$(\pi + \nu/2) < s, \qquad t < -\rho - \nu/2$$

the matrix

$$\begin{pmatrix} D_{XX} & D_{YX} \\ D_{XY} & D_{YY} \end{pmatrix}$$

defines a morphism

$$F : \Gamma^s(X, E_1) \oplus \Gamma^{t+r+\nu/2}(Y, F_1) \to \Gamma^t(X, E_2) \oplus \Gamma^{s-p-\nu/2}(Y, F_2).$$

3. Elliptic morphisms.

Definition 3. We shall call the morphism F elliptic if:

I. The morphism F is elliptic in the usual sense ([1]). This means that its symbol $\sigma(D_{XX}) \in \mathrm{Iso}(\pi^* E_1, \pi^* E_2)$, where $\pi : S^*(X) \to X$ is the fibration over the cosphere bundle; $\pi^* E_i$, $i = 1, 2$, denote the bundles induced by the mapping π.

II. We now formulate the ellipticity condition for the restriction F_Y of the morphism F to the submanifold Y. This condition is local in character; therefore, without loss of generality, we may assume that $X = \mathbb{R}^N$, $Y = \mathbb{R}^n$ $(n + \nu = N)$. Fixing the coefficients of the principal part of the morphism F_Y at some point and making a Fourier transform in the space $\mathbb{R}^n$, we arrive at a family of morphisms

$$F_Y : \Gamma^s(\mathbb{R}^\nu) \oplus \mathbb{C}^l \to \Gamma^t(\mathbb{R}^\nu) \oplus \mathbb{C}^l, \tag{1}$$

parametrized by the unit sphere S^{n-1} in the space $\mathbb{R}^n$. The ellipticity condition for the morphism F_Y consists in the fact that, for each value of the parameter, the morphism (1) is an isomorphism.

This condition can be given an algebraic form. Choose in $\mathbb{R}^N$ coordinates $(x^1, \dots, x^n, t^1, \dots, t^\nu)$, $n + \nu = N$, in such a way that the manifold Y is described by ν linear equations $t^1 = \dots = t^\nu = 0$, or, briefly, $t = 0$. Let $\xi = (\xi_1, \dots, \xi_n)$ and $\tau = (\tau_1, \dots, \tau_\nu)$ be the variables dual to the variables $x = (x^1, \dots, x^n)$ and $t = (t^1, \dots, t^\nu)$, respectively.

Then condition II is equivalent to unique solvability in the space $\Gamma^s(\mathbb{R}^\nu)$, for $\xi \in S^{n-1}$, of the following system of (pseudo)differential equations

$$D_{XX} u + D_{YX} \delta v = 0, \tag{2}$$

$$D_{XY} u\big|_{t=0} + D_{YY} v = g. \tag{3}$$

Having fixed the numbers v, we obtain that the unique solution of equation (2) is the function

$$u = -\int \sigma(D_{XX}^{-1})\,\sigma(D_{YX})\,d\tau \cdot v.$$

From equation (3) we obtain that the numbers v satisfy a linear algebraic system of equations with matrix

$$\left\|\sigma(D_{YY}) - \int \sigma(D_{XY})\,\sigma(D_{XX}^{-1})\,\sigma(D_{YX})\,d\tau\right\|. \tag{4}$$

The condition that the determinant of this matrix be nonzero is equivalent to condition II.

The following theorem establishes a necessary and sufficient condition for the ellipticity of the morphism F.

Proposition 2 (a priori estimate). *Let the morphism F be elliptic. Then, and only then, there exists a constant c, independent of the elements of the function spaces, such that for any $u \in \Gamma^s(X, E_1)$ and any $v \in \Gamma^{t+r+\nu/2}(Y, F_1)$ the inequality holds*

$$\|u\|_s + \|v\|_{-m+r+\nu/2} \le c(\|D_{XX}u\|_{s-m} + \|D_{YY}v\|_{-p-\nu/2} + \|D_{XY}u\|_{s-p-\nu/2} + \|D_{YX}v\|_{s-m} + \|u\|_{s-1} + \|v\|_{-m+r+\nu/2-1}). \tag{5}$$

4. Algebra of morphisms. The set of morphisms is naturally endowed with the structure of a (noncommutative) algebra over the field of complex numbers. Here the composition of morphisms corresponds to the product of the corresponding matrices, while the adjoint morphism F^* corresponds to the transposed matrix with formally adjoint expressions.

As a consequence of this, we obtain that the morphisms F and F^* are elliptic simultaneously.

5. Finiteness theorem. To formulate the main theorem we shall need the following definitions ([3]). Let H_1, H_2 be Banach spaces.

Definition 4. A morphism $F : H_1 \to H_2$ is called **almost invertible on the right** (**on the left**) if there exist morphisms $R_\mathrm{r}, R_\mathrm{l} : H_2 \to H_1$ such that $FR_\mathrm{r} = 1_{H_2} + T_{H_2}$, $R_\mathrm{l}F = 1_{H_1} + T_{H_1}$, where $1_{H_1}, 1_{H_2}$ are the identity maps, and T_{H_1}, T_{H_2} are compact maps of the corresponding spaces.

We can now formulate the main theorem.

Theorem. *The following conditions are equivalent:*

1) *The morphism F is elliptic.*

1*) *The morphism F^* is elliptic.*

2) *The morphism F is almost invertible (on the right and on the left).*

2*) *The morphism F^* is almost invertible (on the left and on the right).*

3) *The morphism F is Fredholm.*

3*) *The morphism F^* is Fredholm.*

4) *There exists a constant c, independent of the elements of the function spaces, such that for any element $f \in H_1$ the inequality holds*

$$\|f\|_1 \leq c(\|Ff\|_2 + \|f\|_0),$$

where the embedding $H_0 \to H_1$ is compact.

4*) *There exists a constant c^*, independent of the elements of the function spaces, such that for any element $\varphi \in H_2^*$ the inequality holds*

$$\|\varphi\|_2^* \leq c^*(\|F^*\varphi\|_1^* + \|\varphi\|_0'),$$

and the embedding $H_0' \to H_2^*$ is compact.

6. Examples. 1) Let in the matrix

$$\begin{pmatrix} D_{XX} & D_{YX} \\ D_{XY} & D_{YY} \end{pmatrix} \tag{6}$$

all operators except the operator D_{XX} be identically equal to zero: $D_{XY} \equiv D_{YX} \equiv D_{YY} \equiv 0$. In this case the elliptic morphism F corresponding to this matrix reduces to an ordinary elliptic operator on the (closed) manifold X.

2) Suppose that in the matrix (6) the operators D_{YX} and D_{XY} are identically equal to zero, while the operator D_{YY} is nonzero: $D_{YY} \neq 0$. In this case the morphism F is represented as the direct sum of two morphisms: $F = D_{XX} \oplus D_{YY}$. The ellipticity condition for the morphism F is in this case the condition of ellipticity of each of the operators D_{XX} and D_{YY}. Moreover, one can show that the identity equality to zero of at least one of the two operators D_{XY} and D_{YX} entails, first, the independence of the kernel and cokernel of the morphism F from the other operator and, second, the ellipticity of the operator D_{YY}.

3) To boundary-value problems in the sense of (3) there corresponds a morphism whose matrix has the following special form:

$$\alpha) \quad D_{YY} = 0,$$

$$\beta) \quad D_{YX} = \partial^j / \partial n^j, \quad |j| \leqslant l.$$

Here n is the field of normal v-frames on the manifold; j is a multi-index: $j = (j_1, \dots, j_l)$, $|j| = \Sigma j_k$,

$$l = \begin{cases} [m - s - v/2], & \text{if } m - s - v/2 \text{ is not an integer,} \\ m - s - v/2 - 1, & \text{if } m - s - v/2 \text{ is an integer.} \end{cases}$$

The matrix (4) is transformed in this case into the matrix

$$\left\| \int \sigma(D_{XY})_j \, \sigma(D_{XX}^{-1}) \, \tau^k \, d\tau \right\|.$$

The nonvanishing, for $\xi \neq 0$, of the determinant of this matrix at a point coincides with the ellipticity condition for the boundary-value problem introduced in (3).

7. **Generalizations.** For lack of space I cannot discuss in detail the possible generalizations of this work. I note only that:

a) the results obtained in the paper are valid for elliptic (co)boundary morphisms defined on a chain of embedded manifolds:

$$X \supset \partial^{v_1} X \supset \partial^{v_2} X \supset \dots;$$

here $\partial^{v_\alpha} X$ is a submanifold of the manifold X of codimension v_α, and $v_1 \leqslant v_2 < \dots$

b) A parabolic theory is constructed in a natural way in the cylinder $X \times I$, where I is the interval $0 \leqslant t \leqslant 1$. It is proved that the corresponding parabolic morphism is an isomorphism, and thus the parabolic (co)boundary-value problem turns out to be uniquely solvable.

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Note: Figure translations are in progress. See original paper for figures.