Sufficient conditions for the absence of periodic trajectories of autonomous systems in the case of multiply connected regions

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Abstract

Two-dimensional autonomous systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{1}$$

are considered, where the right-hand sides have isolated singular points O_j of the pole type. The concept of a "quasi-residue" J_j of a singular point O_j is introduced as the limit value of the integral

$$J_j = \frac{1}{2\pi} \lim_{r\to 0} \oint Q\,dx - P dy \tag{2}$$

taken over a contour $|\gamma|$ surrounding the singular point O_j as the contour $|\gamma|$ deforms into a point (without crossing the singular point O_j). In terms of quasi-residues J_j, sufficient conditions for the absence of periodic trajectories for the system (1) are established. The obtained negative criteria are a known generalization of the Bendixson and Dulac criteria to the case of multiply connected domains, since the phase plane is considered here with excluded points O_j where the poles are located.

Full Text

Preamble

This work, following the foundational principles established in [1, 2], examines the qualitative behavior of autonomous systems of differential equations. We consider a system of the form:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are continuous functions. Associated with this system is the differential form:

$$\omega = Q(x, y)dx - P(x, y)dy \tag{1.2}$$

The equilibrium points $O_j(x_j, y_j)$ of the system (1.1) are defined by the simultaneous vanishing of the functions P and Q, such that $P(x_j, y_j) = Q(x_j, y_j) = 0$. For each such singular point, we define a characteristic index J_j using the line integral of the form (1.2) along a small closed contour γ_j surrounding the point O_j:

$$J_j = \lim_{r\to 0} \frac{1}{2\pi} \oint_{\gamma_j} \frac{Q(x, y)dx - P(x, y)dy}{r^2} \tag{1.3}$$

where $r = \sqrt{(x - x_j)^2 + (y - y_j)^2}$. This integral characterizes the local rotation and divergence properties of the vector field near the singularity.

By applying Green' s theorem to the region D bounded by a contour C and containing several singular points O_j, we can relate the line integral along the boundary to the sum of the indices and the integral of the divergence over the domain:

$$\oint_C \omega = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) dxdy \tag{1.4}$$

This relationship allows for a global qualitative analysis of the phase portrait of the system (1.1).

2. Stability and Divergence Analysis

Let us consider the closed trajectories (limit cycles) of the system (1.1). If a region G contains singular points O_j, the integral along the boundary Γ of the region can be expressed as the sum of the integrals along the small contours γ_i surrounding each point:

$$\oint_\Gamma \omega = \oint_C \omega + \sum \oint_{\gamma_i} \omega$$

Using the divergence of the vector field $\mathbf{V} = P\mathbf{i} + Q\mathbf{j}$, we obtain:

$$\iint_D \mathrm{div}\mathbf{V}\, dxdy = \oint_C (Qdx - Pdy) \tag{2.1}$$

The behavior of the system can be classified based on the value of the integral J_j. Specifically, for a singular point O_j, the following conditions are of interest:

$$\text{a) } J < 0, \text{ b) } J = 0, \text{ c) } J > 0. \tag{2.3}$$

These conditions, combined with the sign of the divergence $\mathrm{div}\mathbf{V} = P_x + Q_y$, determine the stability and the nature of the equilibrium points (e.g., nodes, foci, or saddles).

As an example, consider a linear system:

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy \tag{2.5}$$

For the origin $O(0,0)$, the index J is related to the trace of the matrix, $-\frac{1}{2}(a+d)$. The divergence is given by $\mathrm{div}\mathbf{V} = a + d$. In the specific case where $a = d = 1$ and $b = c = 0$, we find $\mathrm{div}\mathbf{V} = 2$, indicating a source. Conversely, if $a = d = 0$ and $b = -c$, then $J = 0$ and $\mathrm{div}\mathbf{V} = 0$, which corresponds to a center.

3. Generalized Weight Functions

The analysis can be extended by introducing a weight function $X(x, y)$ into the differential form. We define a generalized index J^* as:

$$J^* = \lim_{r \to 0} \oint_\gamma X(x,y)(Qdx - Pdy) \tag{3.1}$$

A common choice for the weight function is:

$$X(x,y) = [(x-x_j)^2 + (y-y_j)^2]^\mu \tag{3.2}$$

where μ is a parameter chosen to ensure the convergence of the integral near the singularity O_j. The generalized divergence of the weighted vector field $\mathbf{V}^* = X\mathbf{V}$ is then:

$$\mathrm{div}\mathbf{V}^* = \nabla \cdot (X\mathbf{V}) = X\mathrm{div}\mathbf{V} + \mathbf{V} \cdot \nabla X \tag{3.5}$$

By analyzing the sign of $\mathrm{div}\mathbf{V}^*$ within a domain D, one can establish criteria for the non-existence of closed trajectories (Bendixson' s criterion and its generalizations). If $\mathrm{div}\mathbf{V}^*$ does not change sign in a simply connected region, the system (1.1) cannot have periodic solutions entirely contained within that region.

References

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5. Lefschetz, S. *Differential Equations: Geometric Theory*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.