DIFFERENTIATION IN LINEAR TOPOLOGICAL SPACES

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DIFFERENTIATION IN LINEAR TOPOLOGICAL SPACES

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The present paper contains an exposition of the theory of differentiation in linear topological spaces (l.t.s.).* Definitions of the first and higher derivatives are introduced, and theorems are formulated on the mean value, on the differentiation of a composite and an inverse function, on the connection between total and partial derivatives, on interchanging the order of differentiation, on finding a primitive, Taylor' s formula is given, and the connection with the theory of variational derivatives is indicated.

A definition of the derivative of a mapping of one l.t.s. (not assumed to be normed) into another for various cases has been proposed by many authors ([1–6, 10, 11]). The definition adopted by us is a development of the definition of Sebastiao e Silva, who introduced, in essence (in our notation), $(X, \beta, \beta; \dots; X, \beta, \beta)$-derivatives.

Let X be a linear space; Y and H l.t.s., $H \subseteq X^{**}$; σ some system of subsets of the space H; β some system of bounded subsets of the space H; $\mathcal{L}(H, Y)$ the linear space of all linear continuous mappings of the space H into Y; $\mathcal{L}_\beta(H, Y)$ the l.t.s. obtained by endowing the space $\mathcal{L}(H, Y)$ with the topology of uniform convergence on the sets of the system β ([7]).

Definition 1. A mapping $f : X \to Y$ will be called σ-differentiable at a point $x \in X$ with respect to the subspace H (briefly, (H, σ)-differentiable at the point x) if there exists an element $f'(x)$ of the space $\mathcal{L}(H, Y)$ such that, for any set $S \in \sigma$,

$$\tau^{-1}(f(x + \tau h) - f(x)) \to f'(x) \cdot h$$

as $\tau \to 0$ ($\tau \in R$) uniformly with respect to h belonging to S. The mapping $f'(x)$ of the space H into Y will be called the σ-derivative of the mapping f at the point x with respect to the subspace H (briefly, the (H, σ)-derivative at the point x). The mapping $f' : x \to f'(x)$ of the space X into $\mathcal{L}_\beta(H, Y)$ will be called the (σ, β)-derivative of the mapping f with respect to the subspace H (briefly, the (H, σ, β)-derivative).

Definition 2. The $(\sigma_1, \beta_1; \dots; \sigma_n, \beta_n)$-derivative $f^{(n)}$ with respect to the subspaces $H_1, \dots, H_n$ (briefly, the $(H_1, \sigma_1, \beta_1; \dots; H_n, \sigma_n, \beta_n)$-derivative) of a mapping $f : X \to Y$ will be called the (H_n, σ_n, β_n)-derivative of the $(H_1, \sigma_1, \beta_1; \dots; H_{n-1}, \sigma_{n-1}, \beta_{n-1})$-derivative $f^{(n-1)}$. The mapping f will be called $(H_1, \sigma_1, \beta_1; \dots; H_{n-1}, \sigma_{n-1}, \beta_{n-1}; H_n, \sigma_n)$-differentiable at the point x if the derivative $f^{(n)}$ is defined at the point x; the value of this derivative at the point x will be called the $(H_1, \sigma_1, \beta_1; \dots; H_{n-1}, \sigma_{n-1}, \beta_{n-1}; H_n, \sigma_n)$-derivative at the point x.

The derivative $f^{(n)}(x)$ we can and shall also regard as a multilinear mapping of the product $H_1 \times \dots \times H_n$ into Y.

We shall call the $(H_1, \sigma_1, \beta_1; \dots; H_n, \sigma_n, \beta_n)$-derivative a **strong** (**weak**) $H_1 \dots H_n$-derivative if each of the systems σ_k, β_k,

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* Everywhere below, separable l.t.s. over the field of real numbers are considered.
** The symbol $H \subseteq X$ means that H is embedded in X as a linear subspace.
*** Here and below we write $f \cdot (x_1, \dots, x_n)$ instead of $f(x_1, \dots, x_n)$, if f is a multilinear mapping of the product of linear spaces $X_1 \times \dots \times X_n$ into Y.

$k = 1, \dots, n$, consists of all bounded (finite) subsets of the space H_k. We shall simply write "derivative" instead of $H_1 \dots H_n$-derivative if $H_1 = \dots = H_n = X$. For functions of a real variable ($f : R^1 \to Y$), weak and strong differentiability coincide.

Define by induction the difference of order n of a mapping $f : X \to Y$ at the point $x \in X$, for increments $h_1, \dots, h_n \in X$, by the equalities

$$\Delta^n f(x; h_1, \dots, h_n) = \Delta^{n-1} f(x+h_n; h_1, \dots, h_{n-1}) - \Delta^{n-1} f(x; h_1, \dots, h_{n-1}), \quad n \geq 2,$$

$$\Delta^1 f(x; h_1) \equiv \Delta f(x; h_1) = f(x + h_1) - f(x).$$

Let A be a subset of a linear topological space E. We shall denote by $\overline{\Gamma A}$ the closed convex hull of A.

Theorem 1 (on the mean). *Let the space Y be locally convex. Then, if (for some $\sigma_k, \beta_k,\ k = 1, \dots, n$) the mapping $f : X \to Y$ is $(H, \sigma_1, \beta_1; \dots; H, \sigma_n)$-differentiable at every point of the set*

$$P = \{x_0 + \theta_1 h_1 + \dots + \theta_{nh} n,\ 0 < \theta_i < 1,\ i = 1, \dots, n\},$$

$x_0 \in X,\ h_1, \dots, h_n \in H$, then

$$\Delta^n f(x_0; h_1, \dots, h_n) \in \overline{\Gamma}\{f^{(n)}(x) \cdot (h_1, \dots, h_n),\ x \in P\}.$$

For $n = 1$, Theorem 1 is a generalization of Lagrange' s theorem on finite increments.

The condition that the space Y be locally convex is essential for the validity of Theorem 1, as the following shows.

Example 1. Let Y be the space of all real measurable functions on the interval $[0,1]$, with the (linear, but not locally convex) topology of convergence in measure. For each $\tau \in R^1$, denote by y_τ the function on $[0,1]$ equal to 1 for $t \leq \tau$ and to zero for $t > \tau$. The derivative of the mapping $f : \tau \to y_\tau$ from the space R^1 into Y is identically equal to zero; however, the mapping f is not constant.

Definition 3. A mapping $f : X \to Y$ will be called **continuous at the point** $x \in X$ **with respect to the subspace** H (in short, H-continuous at the point x) if the mapping $h \to f(x+h)$ from the space H into Y is continuous at zero.

Theorem 2. *If H is a metrizable l.t.s. and the mapping $f : X \to Y$ is strongly H-differentiable at the point $x \in X$, then it is H-continuous at this point.*

The condition that the space H be metrizable is here in fact essential, as the following shows.

Example 2. Let, for each $a > 0$, K_a be the normed space of all real continuous finite functions defined on the real line whose supports are contained in the interval $(-a, a)$, with norm

$$\|x\|_a = \sup_{t \in (-a,a)} |x(t)|;$$

let K be the (strict) inductive limit of the sequence of spaces $K_n,\ n = 1, 2, \ldots$. Consider the countable set S of points (x_{mk}) of the space K ([8]):

$$x_{mk}(t) = m^{-1}\varphi(t) + k^{-1}\varphi(t-m), \quad m, k = 1, 2, \ldots,$$

where $\varphi \in K$. The characteristic function of the set S is strongly differentiable at the point $x = 0$, but is not continuous at this point.

For normed spaces, strong differentiability coincides with Fréchet differentiability.

Theorem 3 (on differentiating an inverse function). *Let X be a metrizable l.t.s., and let $f : X \to Y$ be a homeomorphism of an open set $A \subset X$ onto an open set $B \subset Y$; let g be the inverse homeomorphism. If the mapping f is strongly differentiable at the point $x \in A$ and $f'(x)$ is a linear homeomorphism of the space X onto Y, then the mapping g is strongly differentiable at the point $y = f(x)$ and*

$$g'(y) = (f'(x))^{-1}.$$

Let K, Z be l.t.s., $K \subset Y$.

Theorem 4 (on differentiating a composite function). *Let the mapping $f : X \to Y$ be (H, σ)-differentiable at the point $x \in X$, let the mapping $g : Y \to Z$*

be strongly K-differentiable at the point $f(x)$, and let the compatibility condition be satisfied:

$$x_1 - x_2 \in H, \quad x_1, x_2 \in X \Rightarrow f(x_1) - f(x_2) \in K.$$

Then, if in the space K there exists a countable fundamental system of bounded sets, the composition $(g \circ f) : X \to Z^{-}(H, \sigma)$ is differentiable at the point x and

$$(g \circ f)'(x) = g'(f(x)) \circ f'(x).$$

Theorem 5 (on the relation between total and partial derivatives). Let the space Y be locally convex, and let the space $H \subseteq X$ be the direct sum of two l.t.s. $H = H_1 \oplus H_2$. If for $k = 1, 2$ the (H_k, σ_k, β_k)-derivatives f_k of the mapping $f : X \to Y$ are defined in some H-neighborhood* of the point $x \in X$ and are H-continuous at this point, then the mapping f has at the point x an (H, β)-derivative $f'(x)$, where β is a system of bounded sets in H of the form $B_1 + B_2$, $B_1 \in \beta_1$, $B_2 \in \beta_2$, and

$$f'(x) \cdot h = f_1(x) \cdot h_1 + f_2(x) \cdot h_2,$$

where $h = h_1 + h_2$, $h_1 \in H_1$, $h_2 \in H_2$.

We formulate two theorems on the interchange of the order of differentiation.

Theorem 6 (Young' s theorem). Let the space Y be locally convex, and let the space $H \subseteq X$ be the direct sum of two l.t.s. $H = H_1 \oplus H_2$. If both weak H_1- and H_2-derivatives of the mapping $f : X \to Y$ are strongly H-differentiable at the point $x \in X$, then at this point there exist a weak $H_1 H_2$-derivative $f_{12}(x)$ and a weak $H_2 H_1$-derivative $f_{21}(x)$, and

$$f_{12}(x) \cdot (h_1, h_2) = f_{21}(x) \cdot (h_2, h_1), \qquad h_1 \in H_1,\ h_2 \in H_2.$$

Theorem 7 (Schwarz' s theorem). Let the space Y be locally convex, and let the space $H \subseteq X$ be the direct sum of two l.t.s. $H = H_1 \oplus H_2$. If the mapping $f : X \to Y$ is weakly H-differentiable in an H-neighborhood of the point $x \in X$, and if the weak $H_1 H_2$-derivative f_{12} is H-continuous at the point x, then at this point there exists a weak $H_2 H_1$-derivative $f_{21}(x)$, and

$$f_{12}(x) \cdot (h_1, h_2) = f_{21}(x) \cdot (h_2, h_1) = \lim_{(\tau_1, \tau_2) \to (0,0)} \tau_1^{-1} \tau_2^{-1} \Delta^2 f(x; \tau_1 h_1, \tau_2 h_2).$$

Theorem 8. Let the space Y be locally convex. If the (H, σ, β)-derivative of the mapping $f : X \to Y$ is H-continuous at the point $x \in X$, then at this point the mapping f has an (H, β)-derivative.

Theorem 9 (Taylor' s formula with remainder in Peano form). Let the space Y be locally convex. If the mapping $f : X \to Y$ is $(\underbrace{H, \beta, \beta; \ldots, H, \beta, \beta}_{n})$-differentiable at the point $x \in X$, where β is some system of bounded sets in H

containing all bounded sets of dimension n, then

$$f(x+h)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x)\cdot(\underbrace{h,\dots,h}_{k})+r(x,h,1),$$

where, for any set $B\in\beta$,

$$\tau^{-n}r(x,h,\tau)\to 0\quad\text{as }\tau\to 0$$

uniformly with respect to $h\in B$.

Theorem 10 (generalized Taylor formula with integral term). Let X be a complete l.t.s. possessing a countable fundamental system of bounded sets, Y a Fréchet space, and α a mapping of the real line R^1 into X. If the mapping α is n times differentiable at every point of the interval $[0,t]$ and the derivative $\alpha^{(n)}$ is linear[9] on $[0,t]$, and the mapping f is n times strongly differentiable in some neighborhood of the image of the segment $[0,t]$ under the mapping α and the derivative $f^{(n)}$ is continuous at every point of this neighborhood, then

$$f(x_t)=f(x_0)+\sum_{k=1}^{n}\frac{1}{k!}f^{(k)}(x_0)\cdot\sum_{p=k}^{n-1}\sum_{l_1+\dots+l_k=p}t^p b(l_1,\dots,l_k)(\alpha^{(l_1)}(0),\dots$$
$$\dots,\alpha^{(l_k)}(0))+\int_0^t\frac{(t-\tau)^{n-1}}{(n-1)!}\sum_{k=1}^{n}f^{(k)}(\alpha(\tau))\cdot\sum_{l_1+\dots+l_k=n}c(l_1,\dots,l_k)\cdot$$
$$\cdot(\alpha^{(l_1)}(\tau),\dots,\alpha^{(l_k)}(\tau))\,d\tau.$$

* We call H-neighborhoods of the point x sets of the form $x+U$, where U is a neighborhood of zero in H.

where $x_t\equiv a(t)$,

$$b(l_1,\dots,l_k)=\frac{k!}{(l_1+\dots+l_k)!}\,c(l_1,\dots,l_k)=k!\left(\prod_{i=1}^{k}l_i!\prod_{j=1}^{n}m_j!\right)^{-1},$$

m_j is the number of indices $l_1,\dots,l_k$ equal to j.

Here the integral is understood in the Riemann sense.

Definition 3. Let φ be a mapping of the interval $[0,1]$ into the linear space X; let σ be a system of subsets of the space X. We shall call the mapping φ a **σ-Lipschitz curve** in X if there exists a set $S\in\sigma$ such that $\varphi(\tau_1)-\varphi(\tau_2)\in(\tau_1-\tau_2)S$ for all $\tau_1,\tau_2\in[0,1]$.

Theorem 11 (on finding a primitive). Let X be an l.t.s., Y a sequentially complete locally convex space; let σ be some system of subsets of X containing all bounded sets of dimension two*; let A be a connected open set in X such that every closed polygonal line (with a finite number of links) lying entirely in

A is homologous to zero in A; let $\mathfrak{A}$ be some class of β-Lipschitz curves in A, containing all polygonal lines lying entirely in A, and let $x_0 \in A$, $y_0 \in Y$. Then, if the mapping $g : X \to \mathcal{L}_\beta(X, Y)$ is continuous and σ-differentiable at every point of the set A, and the derivative $g'(x)$ is symmetric** for all $x \in A$, then there exists a mapping $f : X \to Y$, β-differentiable at every point $x \in A$, such that $f'(x) = g(x)$ at every point $x \in A$, $f(x_0) = y_0$.

This mapping is given by the formula

$$f(x) = y_0 + \int_0^1 g(\varphi(\tau))\, d\varphi(\tau),$$

where φ is any curve from the class $\mathfrak{A}$ joining the points x_0 and x ($\varphi(0) = x_0$, $\varphi(1) = x$), and the integral is understood in the Riemann–Stieltjes sense.

Let us note in conclusion that the theory of the so-called variational derivatives can be constructed as a theory of derivatives of real-valued functionals defined on spaces of functions, with respect to subspaces of "basic functions"; in this case the variational derivatives (of first and, respectively, higher orders) turn out to be mappings from the original space of functions into the space of "generalized functions" (of one ([4]) and, respectively, several variables).

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* By the dimension of a set in a linear space we mean the dimension of the minimal linear manifold containing this set.

** We call an element a of the space $\mathcal{L}(X, \mathcal{L}(X, Y))$ **symmetric** if

$$(a \cdot h_1) \cdot h_2 = (a \cdot h_2) \cdot h_1$$

for all $h_1, h_2 \in X$.

Note: Figure translations are in progress. See original paper for figures.