Reports of the Academy of Sciences of the USSR

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1967. Volume 175, No. 1

UDC 517.54

MATHEMATICS

A. V. SYCHEV

ON SPATIAL QUASICONFORMAL MAPPINGS SATISFYING A HÖLDER CONDITION AT BOUNDARY POINTS

(Presented by Academician M. A. Lavrent' ev on 10 IX 1966)

In papers ([3],[4],[6],[7]) it was proved that every Q-quasiconformal mapping ([2]) of a domain D of three-dimensional Euclidean space R^3 satisfies a Hölder condition on any closed subdomain $\overline{D'}$, $\overline{D'} \subset D$. In the present note the fulfillment of this condition is established for Q-quasiconformal mappings of certain classes of domains also at boundary points.

For a given set $E \subset R^3$, ∂E denotes its boundary, $\overline{E}$ its closure, $m(E)$ the three-dimensional Lebesgue measure, and for given sets $E_1, E_2 \subset R^3$, $r(E_1, E_2)$ the distance between them. Further, for a finite point P and $t > 0$, $B^3(P,t)$ denotes the ball $|x - P| < t$, and $S^2(P,t)$ its boundary sphere; if $P = 0$, the following abbreviations are also used:

$$B^3(t) = B^3(0,t), \quad B^3 = B^3(1), \quad S^2(t) = S^2(0,t), \quad S^2 = S^2.$$

A bounded domain D is called a **ring domain** if it is homeomorphic to the domain enclosed between concentric spheres.

Let D be a ring domain; let B_0 and B_1 be its inner and outer boundary components; and let $F_0 \subset B_0$ and $F_1 \subset B_1$ be certain distinguished simply connected boundary continua.

We shall say that a curve $\gamma \subset \overline{D}$ **connects** F_0 and F_1 in D if $\gamma \cap F_0 \neq \emptyset$, $\gamma \cap F_1 \neq \emptyset$, where $\emptyset$ is the empty set. We shall also say that a surface $\sigma \subset \overline{D}$ **separates** F_0 and F_1 in D if every curve connecting F_0 and F_1 in D intersects it in at least one point.

Lemma. Let Σ be a certain family of surfaces contained in the ball B^3; let Σ^* be the image of Σ under the transformation $y = x/|x|^2$, and let Σ_1 be the

family of surfaces $\sigma_1 = \sigma \cup \sigma^*$, $\sigma \in \Sigma$, $\sigma^* = y(\sigma) \in \Sigma^*$.
Then

$$M(\Sigma) = \sqrt{2}\, M(\Sigma_1). \tag{1}$$

We shall say that a domain D is **a-locally simply connected at the boundary point** P if one can indicate an $a > 0$ such that $\partial D \cap B^3(P, t)$ is simply connected for all $t \leq a$. If D is a-locally simply connected at every boundary point, then we shall say that it is **a-locally simply connected on the boundary**.

Theorem 1. Let $y = f(x)$, $f(0) = 0$, be a Q-quasiconformal mapping of a bounded domain D, a-locally simply connected at the point $P \in \partial D$ ($a < R_0/2$), onto the ball B^3. Then for every point $Q \in \overline{D} \cap B^3(P, a)$

$$|f(P) - f(Q)| < C|P - Q|^{1/K(Q)}, \tag{2}$$

where $C = C(R_0, Q)$, $K(Q) = Q\sqrt{2}$, $R_0 = r(0, \partial D)$.

Proof. In view of the a-local simple connectedness of the domain D at the point P, for every point $Q \in \overline{D} \cap B^3(P, a)$ there exists a curve $\gamma_{P,Q}$, connecting P and Q, such that: a) $\gamma_{P,Q} - Q \subset \partial D \cap B^3(P, |P - Q|)$, if $Q \in \partial D$; b) $\gamma_{P,Q} - P - Q \subset D \cap B^3(P, |P - Q|)$, if $Q \in D$.

Let Q_0 be a point for which $|Q_0| < R_0/2$. Denote $r = |Q_0|$, $r^* = |f(Q_0)|$, $\tilde{r} = |P - Q|$, $\tilde{r}^* = |f(P) - f(Q)|$; $F_0 = [0, Q_0]$, $F_0^* = f(F_0)$, $F_1 = \gamma_{P,Q}$, $F_1^* = f(F_1)$. Then $D - (F_0 \cup F_1)$ and $B^3 - (F_0^* \cup F_1^*)$ are ring domains with boundary components $B_0 = F_0$, $B_1 = \partial D \cup F_1$ and $B_0^* = F_0^*$, $B_1^* = S^2 \cup F_1^*$, respectively. Let Σ and Σ^* be the families of surfaces separating F_0, F_1 in $D - (F_0 \cup F_1)$ and F_0^*, F_1^* in $B^3 - (F_0^* \cup F_1^*)$.

By the Q-quasiconformality of $f(x)$,

$$M(\Sigma) \leq Q M(\Sigma^*). \tag{3}$$

Using the known properties of moduli ([1]), Theorem 4 ([5]), and ([1]), we obtain the estimates

$$M^*(\Sigma) < \frac{2}{2\sqrt{\pi}} \ln \frac{R_0^2}{2r\tilde{r}}, \qquad M(\Sigma^*) < \frac{V^2}{2\sqrt{\pi}} \ln \frac{2\lambda^2}{r^*\tilde{r}^*}, \qquad 4 \leq \lambda \leq 1, 2, 4, \dots,$$

whose substitution in (3) gives (2).

Corollary 1. *The family $\{f(x)\}$ of normalized Q-quasiconformal mappings of a bounded domain D with an a-locally simply connected boundary onto the ball B^3 is equicontinuous in $\overline{D}$.*

Let $y = \varphi(x)$ be a homeomorphism of the domain $D \subset R^3$; we shall call $\varphi(x)$ a C-isometry, $1 \leq C < \infty$ (a Lipschitz mapping with constant C), if

$$C^{-1}|P_1 - P_2| \leq |f(P_1) - f(P_2)| \leq C|P_1 - P_2|$$

for all $P_1, P_2 \in D$.

We shall say that a domain $D \ni 0$, homeomorphic to a ball, satisfies at the point $P \in \partial D$ the condition (a, C), if one can indicate a sufficiently small $a > 0$ such that: 1) there exists a C-isometry φ_P mapping $D \cap B^3(P, a)$ onto a half-ball H, under which $\partial D \cap B^3(P, a)$ goes into the flat part ∂H, and the point P goes to the center of H; 2) the area of any surface separating $S^2(a)$ and $D \cap S^2(P, a)$ in $D - (\overline{B^3(a)} \cup \overline{B^3(P, a)})$ is not less than the area of $D \cap S^2(P, a)$.

Theorem 2. *Let $y = f(x)$, $f(0) = 0$, be a Q-quasiconformal mapping of a bounded domain D, homeomorphic to a ball, onto a bounded domain D^*, and suppose that the domains D and D^*, at the boundary points P and $P^* = f(P)$, satisfy the condition (a, C). Then for every point $Q \in \overline{D}$, sufficiently close to P,*

$$C_1|P - Q|^{K(C,Q)} < |f(P) - f(Q)| < C_2|P - Q|^{1/K(C,Q)},$$

where $C_1 = C_1(a, C, R_0, R_0^*, V, Q)$, $C_2 = C_2(a, C, R_0, R_0^*, V^*, Q)$, $K(C, Q) = C^2 Q\sqrt{2}$, $R_0 = r(0, \partial D)$, $R_0^* = r(0, \partial D^*)$, $V = m(D)$, $V^* = m(D^*)$.

The proof of Theorem 2 is analogous to the proof of Theorem 1.

Corollary 2. *The family $\{f(x)\}$ of normalized Q-quasiconformal mappings of a bounded domain D, homeomorphic to a ball and satisfying the condition (a, C) on the boundary, onto a domain D^* of the same kind is equicontinuous in $\overline{D}$.*

Institute of Mathematics
Siberian Branch
Academy of Sciences of the USSR

Received
1 IX 1966

REFERENCES

1. B. V. Shabat, DAN, **130**, 1210 (1960).

2. B. V. Shabat, DAN, **132**, 1045 (1960).

3. M. A. Kreines, Matem. sborn., **9** (51), 713 (1941).

4. Yu. G. Reshetnyak, DAN, **130**, 507 (1960).

5. F. W. Gehring, Trans. Am. Math. Soc., **101**, No. 3, 499 (1961).

6. F. W. Gehring, Trans. Am. Math. Soc., **103**, No. 3, 353 (1962).

7. E. D. Callender, Pacific J. Math., **10**, No. 2, 449 (1960).

Note: Figure translations are in progress. See original paper for figures.