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Abstract

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A FINITE-DIFFERENCE METHOD FOR SOLVING DEGENERATE ELLIPTIC AND PARABOLIC EQUATIONS

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In a bounded domain $\Omega \subset E_n$ $(x_n \geq 0)$, one seeks a solution $u(x)$ of the elliptic equation

$$\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + b_i \frac{\partial u}{\partial x_i} - au = f(x), \tag{1}$$

satisfying on the boundary S of the domain Ω

$$S = S_1 \cup S_2, \qquad S_1 \subset (x_n = 0), \qquad S_2 \subset (x_n > 0)$$

one of the conditions:

$$u\big|_S = 0 \tag{2}$$

or

$$|u|\big|_{S_1} < +\infty, \qquad u\big|_{S_2} = 0, \tag{3}$$

depending on the nature of the degeneration of the equation on S_1.

We assume:

$$\nu x_n^{\beta_i} \xi_i^2 \leq a_{ij} \xi_i \xi_j \leq \mu x_n^{\beta_i} \xi_i^2; \qquad \nu, \mu = \text{const} > 0. \tag{4}$$

Then, for $0 \leq \beta_n < 1$, or for $\beta_n \geq 1$ and $b_n \leq -\theta < 0$, $\theta = \text{const} > 0$, the function $u(x)$ must satisfy condition (2), while in the case $\beta_n > 1$ and $b_n \geq -c^2 x_n^{\beta_n - 1 + \varepsilon}$ $(\varepsilon > 0)$, it must satisfy condition (3). We call problem (1)–(2) with $0 \leq \beta_n < 1$ **problem A**, with $\beta_n \geq 1$ **problem B**, and problem (1), (3) **problem C**. To find their approximate solution, we partition the space E_n, by planes parallel to the coordinate axes, into elementary cubes of edge length h in such a way that one of the planes participating in the partition coincides with $x_n = 0$. At the interior points of the mesh domain $\Omega_h \subset \Omega$ we write the difference scheme

$$\gamma \left(a_{ij}^h u_{\bar{x}i}^h\right)_{x_j} + (1-\gamma)\left(a_{ij}^h u_{x_i}^h\right)_{\bar{x}j} + a_i b_i^h u_{x_i}^h + (1-\alpha_i) b_i^h u_{\bar{x}i}^h + a u^h = f^h, \quad (5)$$

where

$$\gamma = \begin{cases} 1 & \text{for problems A, B;} \\ 0 & \text{for problem C,} \end{cases} \qquad \alpha_i = \begin{cases} 1, & b_i^h \geq 0; \\ 0, & b_i^h \leq 0; \end{cases}$$

$$a_{ij}^h(x_1, \dots, x_n) = \begin{cases} \overset{+n}{a}_{ij}, & x_n \geq h; \\ a_{ij}, & x_n = 0; \end{cases} \qquad b_i^h(x_1, \dots, x_n) = \begin{cases} b_i(x), & x_n > h; \\ \overset{-n}{b}_i(x), & x_n = h; \end{cases}$$

$$f^h(x_1, \dots, x_n) = \frac{1}{h^n} \int_{x_1}^{x_1+h} \cdots \int_{x_n}^{x_n+h} f(\xi)\, d\xi;$$

$$\overset{\mp i}{v}(x_1, \dots, x_n) = v(x_1, \dots, x_i \pm h, \dots, x_n);$$

u_{x_i}, $u_{\bar{x}i}$ are the forward and backward difference quotients. In addition to equations (5), on the boundary S_h of the domain Ω_h ($S_h = S_{1h} \cup S_{2h}$, $S_{1h} \subset S_1$) one imposes

boundary conditions (respectively for problems A, B ($\gamma = 1$)):

$$u\,\Big|_{S_h} = 0, \quad (6)$$

and for problem C ($\gamma = 0$)

$$u\,\Big|_{S_{2h}} = 0. \quad (7)$$

Equations (5), (6) for $\gamma = 1$ and (5), (7) for $\gamma = 0$ constitute a system of linear algebraic equations; the number of equations coincides with the number of unknowns. If one assumes that in Ω

$$\underset{\Omega}{\text{vrai}\max} \left\{ a + \frac{1}{2}\frac{\partial b_i}{\partial x_i} \right\} < 0, \quad (8)$$

then, according to the method of paper (1), one obtains the basic estimate:

$$h^n \sum_{\Omega_h} \left[(x_n + h)^{\beta_i} (u^h_{x_j})^2 + (u^h)^2 \right] \le c \|f\|^2_{L_2(\Omega)}, \tag{9}$$

from which follow the uniqueness of the solution of the difference scheme and its convergence to the exact solution of the original problem. Denote by $\dot{D} \cap L_2(\Omega)$ the space of functions (see (2)), obtained from smooth functions vanishing near S, by completion in the norm

$$\|u\|_{\dot{D}\cap L_2(\Omega)} = \int_\Omega \left(x_n^{\beta_i} \left(\frac{du}{dx_i} \right)^2 + u^2 \right) dx.$$

By a generalized solution of problems A, C we mean a function $u(x) \in \dot{D} \cap L_2(\Omega)$ satisfying, for every $\Phi \in \dot{D} \cap L_2(\Omega)$ that vanishes near S_1, the integral identity

$$\int_\Omega \left\{ a_{ij} \frac{\partial u}{\partial x_i} \cdot \frac{\partial \Phi}{\partial x_j} - u \frac{\partial (b_i \Phi)}{\partial x_i} - au\Phi - f\Phi \right\} dx = 0.$$

Analogously to (2), under the assumptions made, the uniqueness of the generalized solution of problems A and C is proved. As in paper (1), one establishes:

Theorem 1. *Let in (1) the coefficients* $a_{ij} \in C(\Omega)$ *and satisfy conditions (4),* $a \in C(\Omega)$, $b_i \in C_1(\Omega)$, $f \in L_2(\Omega)$, *and let condition (8) be fulfilled. Then, if the coefficients* $b_i(x)$ *do not change sign near* S_1*, the polygonal interpolation* $(u^h)'$ *of the mesh solution* u^h*, as* $h \to 0$*, converges weakly in the norm* $\dot{D} \cap L_2(\Omega)$ *to the generalized solution of the original problem.*

If it is known that the desired solution of the original problem is sufficiently smooth, then, analogously to (3), one obtains:

Theorem 2. *If the conditions of Theorem 1 are fulfilled and the solution of the original problem belongs to the space $W_2^2(\Omega)$**, *then the solution of the difference equation converges to the exact solution in such a way that*

$$\|(u^h)' - u\|_{\dot{D}\cap L_2(\Omega)} \le c \left\{ h^{1/2} + \sum_{k=1}^{n} \left(\int_\Omega \left[\sum_{i,j=1}^{n} (a_{ij} - \overset{-k}{a}_{ij})^2 + \right.\right.\right.$$

$$\left.\left.\left. + (b_i - \overset{-k}{b}_i)^2 + (f - \overset{-k}{f})^2 + (a - \overset{-k}{a})^2 \right] dx \right)^{1/2} \right\}. \tag{10}$$

In the particular case of problems where in (1) $a_{ij} = 0$ for $i \ne j$, for u^h the estimate is valid

$$\max_{\Omega_h} |u^h| \leq \max_{\Omega} |f(x)/c(x)|, \tag{11}$$

which is established exactly as is done for difference analogues of nondegenerate equations. From (11), for $u \in C_3(\Omega)$ and smooth

* This condition is not required if at least one of the $\beta_i < 1$ $(i = 1, \ldots, n)$.

** Fulfillment of this condition is guaranteed, for example, when S, the coefficients (1), $b_n|_S < -\theta < 0$, $\beta_n >$

$(\theta = \text{const} > 0)$ or $b_n|_{S_1} > \theta > 0$, $\beta_n > 1$ (see also ([4],[5])).

*** In ([3]) the estimate contains $h^{1/4-\varepsilon}$, $\varepsilon > 0$. However, by the method of ([3]) one obtains directly also the est

coefficients (1), in the usual way (see (6)) one derives the inequality

$$\max_{\Omega_h} |u - u^h| \leq ch. \tag{12}$$

By shifting the grid by $h/2$ relative to the coordinate axes and interpolating the boundary conditions with order h^2, one can obtain a difference scheme for whose solution the quantity on the right-hand side of (12) is $\sim ch^2$.

Let us now consider in $Q = \Omega \times [0, T]$ the parabolic equation

$$-q(x,t)\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_i} a_{ij}(x,t)\frac{\partial u}{\partial x_j} + b_i(x,t)\frac{\partial u}{\partial x_i} + a(x,t)u = f(x,t), \tag{13}$$

where $q(x,t) = t^\alpha d(x,t)$, $d(x,t) > \theta$; $\alpha \geq 0$; $\theta > 0$; the coefficients $a_{ij}(x,t)$ may vanish on $S_1' = S_1 \times [0,T]$. If for $t \in [0,T]$ condition (4) is satisfied, where the β_i are now smooth functions of t, then, quite analogously to the case of an elliptic equation, for each $t \in [0,T]$ one of the boundary conditions, either (2) or (3), is imposed for the function $u(x,t)$ on S_1. In addition, for $\alpha < 1$ the solution must satisfy the initial condition $u|_{t=0} = \varphi(x)$, while for $\alpha > 1$, $|u|_{t=0} < +\infty$.

Let us write in $Q_h = \Omega_h \times [\Delta t, T]$ the difference scheme for equation (13) in the form

$$-qu^h_{\bar t} + \gamma(t)(a^h_{ij}u^h_{\bar x_i})_{x_j} + (1-\gamma(t))(a^h_{ij}u^h_{x_i})_{\bar x_j} + \alpha_i b^h_i u^h_{x_i} + (1-\alpha_1)b^h_i u^h_{\bar x_i} + au^h + f^h = 0, \tag{14}$$

where $u_{\bar{t}}$ is the backward difference with respect to t, and the coefficients a_{ij}^h, b_i^h, a, f^h, γ, α_i are computed at each t in the same way as in the corresponding case of the elliptic equation. In addition, depending on the formulation of the original boundary-value problem, for a given t one should add to (14) the boundary conditions either

$$u^h\big|_{x\in S_h} = 0 \qquad (\text{for } \gamma(t) = 1), \tag{15}$$

or

$$u^h\big|_{x\in S_{2h}} = 0 \qquad (\text{for } \gamma(t) = 0), \tag{16}$$

and in the case $\alpha < 1$ the initial condition

$$u^h\big|_{t=0} = \varphi(x). \tag{17}$$

We assume that the coefficients of (13) for $\alpha > 1$ are such that they can be continuously extended to $t = 0$. Then for $t = 0$ we write the difference scheme (14), setting in it $q(0)u_{\bar{t}}^h(x,0) = 0$. If now to the equations obtained on the layer $t = 0$ we add, depending on the character of the degeneration on S_1 of equation (13) at $t = 0$, either condition (15) or condition (16), then for $t = 0$ we obtain a linear algebraic system with the number of unknowns equal to the number of equations.

If we assume that in the case $\alpha < 1$ condition (8) is satisfied, and for $\alpha > 1$, in some small neighborhood $0 \le t \le \delta$, $\delta > 0$, along with (4),

$$\max_{\Omega,\, 0\le t\le\delta} \left\{ a + \frac{1}{2}\frac{\partial b_i}{\partial x_i} + \left|\frac{\partial q}{\partial t}\right| \right\} < 0, \tag{18}$$

then, analogously to (9), the estimate is established

$$(\Delta t)h^n \sum_{Q_h} \left[(\Delta t)(q u_{\bar{t}}^h)^2 + (x_n + h)^{\beta_i}(u_{x_i}^h)^2 + (u_{\bar{x}_i}^h)^2\right]_{t=\delta} + h^n \sum_{\Omega_h} (q u^h)^2\big|_{t=\delta} \le$$

$$\le c\left[(\Delta t)h^n \sum_{Q_h} (f^h)^2 + h^n \sum_{\Omega_h} (\psi^h)^2\right]; \tag{19}$$

$$\psi^h(x) = \begin{cases} \varphi(x), & \text{for } \alpha < 1, \\ f^h(x,0), & \text{for } \alpha > 1. \end{cases}$$

from which one can prove the uniqueness of the solution of the difference scheme and convergence to the generalized solution. In the case of a smooth solution one can derive an estimate of the rate of convergence similar to (10).

For a particular class of problems, when in (2) $a_{ij} = 0$ for $i \neq j$, boundedness estimates are proved in the maximum norm of the type (11), (12).

In this same case, if one assumes that the behavior of the coefficients a_{ii} and b_i for $0 \leq t \leq T$ is such that the formulation of the boundary condition on S_1, either (2) or (3), does not change in time, β_i do not depend on t, and also that the coefficients are sufficiently smooth, $a_{ii} \in C_2(\overline{Q})$; a, b_i, f, $q(t) \in C_1(\overline{Q})$, and the solution $u \in C_3(\overline{Q})$, then by a method analogous to (7) one proves the convergence of a locally one-dimensional difference scheme, which in this case has the form

$$q(x,t)u^h_{\bar{t}_i} = \gamma\big(a^h_{ii}(x,t)u^h_{x_i}(x,t^*)\big)_{x_i} + (1-\gamma)\big(a^h_{ii}(x,t)u^h_{x_i}(x,t^*)\big)_{\overline{x}i} + \\ + \alpha_i b^h_i(x,t)u^h_{x_i}(x,t^*) + (1-\alpha_i)b^h_i(x,t)u^h(x,t^*) + \\ + c(x,t)u^h(x,t^*) = \frac{1}{i}f(x,t), \tag{20}$$

where

$$t^* = t + \frac{\Delta t}{n}i - \Delta t; \qquad u^h_{\bar{t}_i} = \frac{u^h(x,t^*) - u^h(x,t^* - \Delta t/n)}{\Delta t/n};$$

d^h_{ii}, b^h_i, γ, α_i are chosen analogously to the difference schemes (5), (14);

$$u^h(x,t^*)\big|_{x\in S_h} = 0 \quad \text{for } \gamma = 1; \qquad u^h(x,t^*)\big|_{x\in S_{2h}} = 0 \quad \text{for } \gamma = 0. \tag{21}$$

For (20), (21) one obtains the estimate of the rate of convergence

$$\max_{Q_h} \big|u(x,t) - u^h(x,t)\big| \leq c(\Delta t + h). \tag{22}$$

By a small change in the chosen scheme one can also arrange that in the right-hand side of (22) there stands a quantity of order $c(\Delta t + h^2)$.

The scheme (20), (21) can also be regarded as an iteration method for solving the difference equations (5), (7) and (5), (6).

For the difference schemes described in the paper, and also for schemes with variable steps in x and t (the mesh step was taken smaller near the surface of degeneration), a number of computations were carried out at the Computing Center of Leningrad University. They showed good agreement between the exact and approximate results. This was reported in (8).

The difference schemes used in the paper and the theorems proved can be easily extended to the case when the coefficients are less smooth than indicated in the paper, when several types of degeneracy are present simultaneously, and also to the case when the degeneracy occurs along a part of the boundary not lying in the plane $x_n = 0$. For parabolic equations one can also consider other explicit and implicit difference schemes.

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Note: Figure translations are in progress. See original paper for figures.