SOME METRIZATION THEOREMS FOR FEATHERED SPACES

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SOME METRIZATION THEOREMS FOR FEATHERED SPACES

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This work consists of two parts. The aim of the main—the second—part is the proof of Theorem 2.3. We note that all spaces are assumed to be completely regular, unless it is stated precisely which separation axioms are satisfied, and mappings are continuous and single-valued.

1. In the work ([3]) A. V. Arhangel' skii introduced a new class of topological spaces, namely the class of p-spaces (feathered spaces). Feathered spaces possess many remarkable properties; for example, the addition theorem on weight is valid for them.

1.1. Proposition. Let X be a weakly paracompact p-space. If the diagonal $D = \{(x,x) \mid x \in X\}$ is a set of type G_δ in the space $X \times X$, then X has a uniform base.

Proof. By assumption there exists a countable family $\{U_n \mid n = 1, 2, ...\}$ of open subsets of $X \times X$ such that

$$D = \bigcap_{n=1}^{\infty} U_n.$$

For each natural number n put $\gamma_n = \{V_{\beta n} \mid V_{\beta n} \subset X;\ V_{\beta n} \times V_{\beta n} \subset U_n\}$. It is clear that all γ_n are open coverings of the space X. We shall prove that

$$\bigcap_{n=1}^{\infty} \gamma_n x = x$$

for any point $x \in X$. Suppose the contrary: $y \in \bigcap_{n=1}^{\infty} \gamma_n x$ and $y \neq x$. Then for each n there exists an open subset $V_{\beta n}$ of X such that $x, y \in V_{\beta n}$ and $V_{\beta n} \times V_{\beta n} \subset U_n$; hence it follows that

$$(x, y) \in \bigcap_{n=1}^{\infty} U_n,$$

which is impossible, since

$$\bigcap_{n=1}^{\infty} U_n = D \not\ni (x, y).$$

In view of the weak paracompactness of the space X, the conditions of Lemma 8.2 from ([3]) are satisfied; therefore in the space X there is a refining sequence of coverings. Applying P. S. Aleksandrov' s theorem from ([1]), we conclude that X has a uniform base.

1.2. Corollary (Okuyama). Let X be a paracompact p-space. If the diagonal $D = \{(x,x) \mid x \in X\}$ has type G_δ in $X \times X$, then the space X is metrizable.

It is known that if into every open covering of a collectively normal space one can inscribe a σ-discrete closed covering, then this space is paracompact.

1.3. Proposition. If a topological space X has a σ-discrete net*, then the space $X \times X$ also has a σ-discrete net, and every open subset of $X \times X$ is the sum of a countable family of sets closed in $X \times X$.

This is an obvious assertion.

2. The main result of the paper:

2.1. Addition theorem. Let X be a collectively normal p-space, and $X = M_1 \cup M_2$, where M_1 is a space with a countable ba-

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* A system S of subsets of a space X is called a net in X if for any x and Ox—a point and its neighborhood in X—there exists $P \in S$ such that $x \in P \subset Ox$ ([4]).

second, M_2 is the union of a countable set of closed metrizable subspaces F_n of X $(n = 1, 2, ...)$. Then the space X is metrizable.

Proof. Let ω_n be some σ-discrete base of the space F_n (such a base, as is known, exists; see (7)). Since the set F_n is closed in X, the system ω_n is σ-discrete also in the space X. Next, let ω_0 be some countable base of the space M_1. It is easy to prove that the system Ω, σ-discrete in X,

$$\Omega = \bigcup_{k=0}^{\infty} \omega_k$$

forms a net in the space X. For a complete proof of Theorem 2.1 it remains only to prove the following proposition:

2.2. Proposition. *A collectionwise normal p-space with a σ-discrete net is metrizable.**

Proof. Let X be a collectionwise normal p-space, and let

$$\Omega = \{\gamma_n = \{P_{\alpha n} \mid \alpha \in \theta\} : n = 1, 2, ...\}$$

be a net in the space X, each system γ_n being discrete in X. We note that into any open cover ω one can inscribe some closed σ-discrete cover. For this it suffices to take the totality of all elements of Ω contained, together with their closure, in the cover ω. Consequently, the space X is paracompact. Applying now assertions 1.2 and 1.3, we conclude that the space is metrizable. The theorem is proved.

Recall that all spaces complete in the sense of Čech and all locally bicompact spaces belong to the class of p-spaces.

Remark. In general, the following result is true. If a paracompact p-space X can be covered by a locally countable system of subspaces closed in it and metrizable, then X is metrizable.

Theorem 2.1 allows one to prove the following assertion.

2.3. Theorem. *Let a collectionwise normal p-space X be the sum of a countable family of metrizable subspaces M_n $(n = 1, 2, ...)$. If, for every natural number n, the set*

$$\bigcup_{i=1}^{n} M_i$$

is a closed set of type G_δ in X, then the space X is metrizable.

Proof. Put $N_1 = M_1$ and

$$N_n = M_n \setminus \bigcup_{i=1}^{n-1} N_i$$

for $n = 2, 3, ...$ Obviously,

$$\bigcup_{i=1}^{n} N_i = \bigcup_{i=1}^{n} M_i$$

for every integer $n > 0$. Consequently, there exists a countable system $\{\Gamma_{nj} \mid j = 1, 2, ...\}$ of sets open in X such that

$$\bigcap_{j=1}^{\infty} \Gamma_{nj} = \bigcup_{i=1}^{n} N_i.$$

On the basis of Theorem 2.1 it is enough to prove that each set N_n is the sum of a countable family of closed subsets of X. We prove this. By hypothesis, the set N_1 is closed in X. Let now $n > 1$; then put

$$F_{nj} = \bigcup_{i=1}^{n} N_i \cap (X \setminus \Gamma_{(n-1)j}).$$

By construction, the set F_{nj} is closed in X. Since

$$\bigcap_{j=1}^{\infty} \Gamma_{(n-1)j} = \bigcup_{i=1}^{n-1} N_i,$$

we have

$$N_n = \bigcup_{j=1}^{\infty} F_{nj};$$

this completes the proof of Theorem 2.3.

Recalling the well-known example of P. S. Urysohn (see (8), p. 206) of a non-metrizable countable space, we conclude that if the word "feathered" is omitted in the formulations of assertions 1.1, 2.1, 2.2, and 2.3, then they cease to be true.

Theorem 2.2 together with Theorem 5.5 from (3) lead us to the following conclusion:

2.4. Theorem. *Let $f : X \to Y$ be a perfect mapping of a topological space X onto a metric space Y. If the space*

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* This proposition was proved independently of the author by A. V. Arhangel' skii.

that X is the sum of a countable family of closed metrizable subspaces, then it is itself metrizable.

We now give an assertion somewhat more general than Theorem 2.3.

2.5. Proposition. If a weakly paracompact p-space X is the sum of a countable family of closed subspaces F_n $(n = 1, 2, ...)$ with a uniform base, then it itself has a uniform base.

Proof. It is easily proved that if the space X is the union of a countable number of closed subspaces F_n $(n = 1, 2, ...)$ and, for every natural n, every open subset of F_n is of type F_σ, then every open subset of X is the sum of a countable family of sets closed in X. Comparing this assertion with Lemma 2 of (2), we conclude that every closed subset of X is of type G_δ; consequently, for every integer $n > 0$ there exists a countable family $\{\Gamma_{nk} \mid k = 1, 2, ...\}$ of open subsets of X such that

$$\bigcap_{k=1}^{\infty} \Gamma_{nk} = F_n \quad \text{and} \quad \Gamma_{n(k+1)} \subseteq \Gamma_{nk} \quad \text{for } k = 1, 2, \dots .$$

By hypothesis, in F_n there exists a uniform base

$$\lambda_n = \{\gamma_{nk} \mid k = 1, 2, ...\},$$

splitting into a countable set of successively inscribed coverings such that

$$\bigcap_{k=1}^{\infty} \gamma_{nk} x = x \quad \text{for every point } x \in F_n.$$

For each covering γ_{nk} take an open system in X,

$$\tilde{\gamma}_{nk} = \{U_{\alpha nk} \mid \alpha \in \theta\},$$

such that

$$U_{\alpha nk} \subseteq \Gamma_{nk} \setminus \bigcup_{i=1}^{n-1} F_i \quad \text{and} \quad \tilde{\gamma}_{nk} \cap F_n = \gamma_{nk} \cap \left(F_n \setminus \bigcup_{i=1}^{n-1} F_i \right).$$

Put

$$\omega_k = \bigcup_{n=1}^{\infty} \tilde{\gamma}_{nk}.$$

The system ω_k covers the space X. We shall prove that

$$\bigcap_{k=1}^{\infty} \omega_k x = x$$

for every point $x \in X$. Let n_0 be the first natural number for which $x \in F_{n_0}$, and let k_0 be a natural number such that $x \in \Gamma_{nk}$ for $k \geq k_0$ and $n < n_0$. From the construction of the coverings ω_k $(k = 1, 2, ...)$ it follows that

$$\omega_k x = \tilde{\gamma}_{n_0 k} x \quad \text{for } k \geq k_0;$$

hence

$$\bigcap_{k=1}^{\infty} \omega_k x = \bigcap_{k=k_0}^{\infty} \tilde{\gamma}_{n_0 k} x = \bigcap_{k=k_0}^{\infty} \left(\tilde{\gamma}_{n_0 k} x \cap \Gamma_{n_0 k} \right) = \bigcap_{k=k_0}^{\infty} \gamma_{n_0 k} x = \bigcap_{k=1}^{\infty} \gamma_{n_0 k} x = x.$$

In view of the weak paracompactness of the space X, the hypotheses of Lemma 8.2 of (3) are fulfilled; consequently, in the space X there exists a uniform base.

2.6. Proposition. For any normal topological space X, the following two assertions are equivalent:

1) The space X can be condensed onto a metric space.
2) In the space X there exists a countable family of locally finite systems

$$\{\gamma_n = \{V_{n\alpha} \mid \alpha \in \theta\}\ n = 1, 2, ...\}$$

of open subsets of X with the following properties: a) for any pair of indices $n\alpha$ the set $V_{n\alpha}$ is an open set of type F_σ; b) for any $x, y \in X$ $(x \neq y)$ there is a $V_{n\alpha} \in \bigcup_{n=1}^{\infty} \gamma_n$ such that

$$V_{n\alpha} \cap (\{x\} \cup \{y\})$$

consists of one point.

The proof of this, in substance, repeats the well-known construction of Dowker from (5). Proposition 2.6 could have been used as the basis for proving the essential assertions of the second part.

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CITED LITERATURE

1. P. S. Aleksandrov, Bull. Polish Acad. Sci., Ser. Math., **8**, 135 (1960).
2. A. V. Arkhangel' skii, Bull. Polish Acad. Sci., Ser. Math., **8**, 589 (1960).
3. A. V. Arkhangel' skii, Mat. Sb., **57**, 55 (1965).
4. A. V. Arkhangel' skii, DAN, **126**, 239 (1959).
5. C. Dowker, Am. J. Math., **49**, 200 (1947).
6. A. Okuyama, Proc. Japan. Acad., **40**, 176 (1964).
7. R. Bing, Canad. J. Math., **3**, 175 (1951).
8. P. S. Uryson, *Trudy po topologii i drugim oblastyam matematiki*, **1**, Moscow-Leningrad, 1951.

Note: Figure translations are in progress. See original paper for figures.