THE RADON TRANSFORM IN SPACES OF MATRICES AND GRASSMANN MANIFOLDS

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THE RADON TRANSFORM IN SPACES OF MATRICES AND GRASSMANN MANIFOLDS

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In the present paper the classical Radon transform (1–4) is generalized to the case of spaces of rectangular matrices and Grassmann manifolds.

1. Let $\mathfrak{M}_{n,m}$ be the space of real rectangular matrices with n rows and m columns, $n \geq m$. Consider on $\mathfrak{M}_{n,m}$ the first-degree matrix equation (5)

$$(\Xi, X) \equiv \Xi' X = P, \qquad \Xi, X \in \mathfrak{M}_{n,m}, \qquad P \in \mathfrak{M}_{m,m}, \tag{1}$$

where P and Ξ are fixed matrices; Ξ has maximal rank; the prime denotes transposition; multiplication $\cdot$ is matrix multiplication. Equation (1) defines in $\mathfrak{M}_{n,m}$ a plane of dimension $m(n-m)$.* With this plane one may associate the differential form of degree $m(n-m)$

$$\omega = (-1)^{m^2(i_1+i_2+\dots+i_m-(m+1)m/2)} (\det \hat{\Xi})^{-m}\, dx_{j_1} dx_{j_2} \dots dx_{j_{n-m}}, \tag{2}$$

where $\hat{\Xi}$ is a nonsingular submatrix of the matrix Ξ, composed of the $i_1-, i_2-, \dots, i_m$-th rows, $i_1 < i_2 < \dots < i_m$; $dx_{j_1} dx_{j_2} \dots dx_{j_{n-m}}$ is the product of the differentials of the elements of the remaining rows in the matrix X, $j_1 < j_2 < \dots < j_{n-m}$.

For a "good" function $\varphi(X)$ we have the transform

$$\varphi(X) \to \check{\varphi}(\Xi', P) = \int_{(\Xi, X)=P} \varphi(X)\omega = \int_{\mathfrak{M}_{n,m}} \varphi(X)\delta(\Xi' \cdot X - P)\, dX. \tag{3}$$

Below we shall restrict ourselves to functions from the spaces S and K of infinitely differentiable functions rapidly decreasing at infinity together with their derivatives, and of infinitely differentiable finite functions, respectively. For $m = 1$, (3) is the classical Radon transform. We therefore retain this name also for the case $m > 1$, although the integration in (3) is not carried out over a hyperplane.

2. We describe some properties of the function $\check{\varphi}(\Xi', P)$.

a) **Matrix parity:** for all orthogonal matrices O of order m,

$$\check{\varphi}(O\Xi', OP) = \check{\varphi}(\Xi', P).$$

b) **Matrix (determinantal) homogeneity of degree m:** for any nonsingular symmetric matrix A of order m,

$$\check{\varphi}(A\Xi', AP) = (\det A)^{-m}\check{\varphi}(\Xi', P).$$

c) **Convolution transform:** if $\varphi(X) = \varphi_1 * \varphi_2$, then

$$\check{\varphi}(\Xi', P) = \int_{\mathfrak{M}_{m,m}} \check{\varphi}_1(\Xi', M)\check{\varphi}_2(\Xi', P - M)\, dM.$$

d) **Connection with the Fourier transform:** if $\widetilde{\varphi}(\Xi)$ is the Fourier transform of the function $\varphi(X)$, then

$$\check{\varphi}(\Xi', P) = \frac{1}{(2\pi)^{m^2}} \int_{\mathfrak{M}_{m,m}} \widetilde{\varphi}(\Xi Q) e^{-i \operatorname{tr} P'Q}\, dQ.$$

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* Equation (1) does not describe all planes in $\mathfrak{M}_{n,m}$ of the given dimension. The dimension of the manifold of planes of the form (1) is nm, i.e. the dimension of the space $\mathfrak{M}_{n,m}$.

3. With the Radon transform (3) we associate the problem of integral geometry in the sense of (4), consisting in finding an inversion formula for the transform (3) and in describing the classes of functions $\varphi(X)$ and $\check{\varphi}(\Xi', P)$.

Let $V_{n,m} = \{X : X'X = E^{(m)}\}$ ($E^{(m)}$ is the identity matrix of order m) be the Stiefel manifold, which is the homogeneous space O_n/O_{n-m} (O_n is the orthogonal group of order n); dV is the invariant measure on $V_{n,m}$; $C_{n,m}$ is the volume of $V_{n,m}$; $|X| = \sqrt{X'X}$; $\Gamma_m(\delta) = \pi^{\frac{1}{4}m(m-1)}\Gamma(\delta)\Gamma(\delta - 1/2)\cdots\Gamma(\delta - (m - 1)/2)$ is the gamma-function of the cone of symmetric positive definite matrices of order m. To find the inversion formula we use the equality

$$\frac{1}{\pi^{m(n-m)/2}} \int_{V_{n,m}} \frac{\det |V'(X - X_0)|^{\lambda}}{\Gamma_m((\lambda + m)/2)}\, dV = C_{m,m}\frac{\det |X - X_0|^{\lambda}}{\Gamma_m((\lambda + n)/2)}, \tag{4}$$

which is valid for all complex λ, the integral on the left converging for $\operatorname{Re}\lambda > -1$, and for the remaining λ being defined by analytic continuation in λ. For $\lambda = -n$, (4) gives the inversion formulas. For even $n - m$,

$$\varphi(X) = \frac{(-1)^{m(n-m)/2}}{(2\pi)^{m(n-m)} C_{m,m}} \int_{\Xi_{n,m}} D_P^{n-m} \check{\varphi}(\Xi', \Xi' \cdot X)\, d\Xi, \tag{5}$$

where $\Xi_{n,m}$ is a surface in $\mathfrak{M}_{n,m}$, affinely equivalent to $V_{n,m}$:

$$\Xi_{n,m} = \{\Xi : \Xi = VF;\ V \in V_{n,m},\ F = F(V) \in \mathfrak{M}_{m,m}\}; \quad d\Xi = \det|F|^n dV;$$

$D_P = \det \|\partial/\partial p_{ij}\|$, understood in the operator sense.

For arbitrary n and m,

$$\varphi(X) = \frac{\pi^{m(m-n)/2}}{C_{n,m}} \int_{\Xi_{n,m}} \left(\frac{|t_{11}|^{\lambda} |t_{22}|^{\lambda+1} \dots |t_{mm}|^{\lambda+m-1}}{\Gamma_m((\lambda+m)/2)} \bigg|_{\lambda=-n}, \check{\varphi}(\Xi', T - \Xi' \cdot X) \right)_{T_m} d\Xi, \tag{5'}$$

where T_m is the space of upper triangular matrices of order m, regarded as a Euclidean space of dimension $m(m+1)/2$; $|t_{ii}|^{\mu}/\Gamma((\mu+1)/2)$ $(i = 1, 2, \dots, m)$ are one-dimensional generalized functions, well known for all values of μ (^{2}).

For all $n \geq m > 1$, formula $(5')$ is not local. For $m = 1$ it turns into two classical formulas, depending on the parity or oddness of n.

Further, on the basis of property c), analogues of the Plancherel formula are easily established. We give the case of even $n - m$:

$$\int_{\mathfrak{M}_{n,m}} \varphi(X) \overline{\psi(X)}\, dX =$$

$$= \frac{(-1)^{m(n-m)/2}}{(2\pi)^{m(n-m)} C_{m,m}} \int_{\Xi_{n,m}} \left(\int_{\mathfrak{M}_{m,m}} \check{\varphi}(\Xi', P)\, \overline{D_P^{n-m} \psi(\Xi', P)}\, dP \right) d\Xi. \tag{6}$$

Finally, using the connection between the Radon and Fourier transforms, we establish an analogue of the Paley-Wiener theorem. Namely:

In order that $\check{\varphi}(\Xi', P)$ be the Radon transform of some function $\varphi(X)$ from S or K, it is necessary and sufficient that the following conditions be fulfilled:

1) $\check{\varphi}(A\Xi', AP) = \det|A|^{-m} \check{\varphi}(\Xi', P)$ for any nonsingular $A \in \mathfrak{M}_{m,m}$;
2) $\check{\varphi}(\Xi', P)$ is infinitely differentiable with respect to Ξ when $\det|\Xi| \neq 0$;
3) for fixed Ξ, $\check{\varphi}(\Xi', P)$ is a function of P belonging to the space S or K, respectively;
4) for any $k_1, k_2, \dots, k_m = 0, 1, \dots$ the integral

$$\int_{\mathfrak{M}_{m,m}} \check{\varphi}(\Xi', P) \prod_{i=1}^{m} p_{ii}^{k_i}\, dP$$

is a polynomial in Ξ, homogeneous of degree k_i $(i = 1, \ldots, m)$ in the elements of the i-th column of the matrix Ξ.

4. An analogous theory can be constructed for the case of the space $\mathfrak{N}_{n,m}$ of complex matrices. We note only the presence of two variants of the inversion formula. If

$$\check{\varphi}(H', S) = \int_{\mathfrak{N}_{n,m}} \varphi(Z)\delta(H' \cdot Z - S)\, dZ, \tag{7}$$

then the first formula is as follows:

$$\varphi(Z) = \frac{(-1)^{nm-\frac{1}{2}m(m+1)}}{\pi^{2m(n-m)}\Gamma_{(m)}(m)} \times$$

$$\times \int_H \left(\delta^{(n-1,n-1)}(t_{11}) \cdots \delta^{(n-m,n-m)}(t_{mm}),\ \check{\varphi}(H', T - H' \cdot Z)\right)_{T_m} dH, \tag{8}$$

where T_m is the space of complex upper triangular matrices of order m; $\Gamma_{(m)}(\lambda) = \pi^{\frac{1}{2}m(m-1)}\Gamma(\lambda)\Gamma(\lambda-1)\cdots\Gamma(\lambda-m+1)$ is the gamma function of the cone of Hermitian positive definite matrices of order m; H is a "surface" in $\mathfrak{N}_{n,m}$ containing one representative from each class of matrices representing the complex Grassmann manifold (see the construction of the analogous "surface" in (4)).

The second inversion formula has the form

$$\varphi(Z) = \frac{(-1)^{m(n-m)}}{\pi^{2m(n-m)}} \int_H \det \|\partial_s' \partial_s\|^{m-n} \check{\varphi}(H', H' \cdot Z)\, dH, \tag{9}$$

where $\partial_s = \|\partial/\partial s_{ij}\|$, and $\det \|\partial_s' \partial_s\|$ is understood in the operator sense. For $m = 1$ formulas (8) and (9) coincide.

5. We now consider the real Grassmann manifold $G_{n,m}$ (the set of m-dimensional planes in an n-dimensional vector space passing through the origin). It can be interpreted as the space of rectangular matrices of type $n \times m$, specified up to multiplication on the right by an arbitrary nonsingular matrix of order m.

The Grassmann manifold may be regarded as the homogeneous compact symmetric space $O_n/O_{n-m} \times O_m$. A function $\varphi(X)$, where X is a matrix of type

$n \times m$, will be called a function on $G_{n,m}$ if, for every nonsingular matrix B of type $m \times n$, $\varphi(XB) = \varphi(X)$.

For a bounded summable function $\varphi(X)$ on $G_{n,m}$, $n \geq 2m$, consider the transformation

$$\varphi(X) \to \check{\varphi}(M) = \int_{G_{n,m}} \varphi(X)\delta(M' \cdot X)\, dX, \tag{10}$$

where M is a fixed matrix from $G_{n,m}$ of maximal rank; the integral over $G_{n,m}$ is understood as the integral over $V_{n,m}$ divided by $C_{m,m}$. The equation $M' \cdot X = 0$ defines in $G_{n,m}$ a submanifold $G_{n-m,m}$. The formulas restoring $\varphi(X)$ from its known integrals over the submanifolds $G_{n-m,m}$ have the form

$$\varphi(X) = (2\pi i)^{m(2m-n)} \int_{X' \cdot M=0} D_P^{n-2m} \left\{ \int_{M' \cdot Y=0} \varphi(Y)\omega_M \right\} \omega_X, \tag{11}$$

where $N = X' \cdot M$, the forms ω_X, ω_M are defined analogously to (2), and

$$\varphi(X) = \frac{(-1)^{m(n-2m-1)/2}\Gamma_m((m+1)/2)}{2^{2m(n-2m)-1}\pi^{m(n-2m)}\Gamma_m(m/2)} \times$$

$$\times \int_{T'T<E^{(m)}} \frac{|t_1^1|^{\lambda}}{\Gamma((\lambda+1)/2)} \cdots \frac{|t_m^m|^{\lambda+m-1}}{\Gamma((\lambda+m)/2)} \bigg|_{\lambda=-m-1} \times$$

$$\times \det(E^{(m)} - T'T)^{(n-2m-1)/2} D_T^{n-2m-1} \left\{ \int_{(M,X)=T} \check{\varphi}(M)\omega_X \right\} dT, \tag{11'}$$

where the integration is carried out over the domain of upper triangular matrices T such that the matrix $E^{(m)} - T'T$ is positive definite.

In conclusion we note that the analogous problem is also solved for the complex Grassmann manifold $H_{n,m}$, where, corresponding to (8) and (9), there are likewise two versions of the inversion formula. We give the formula analogous to (9) and (11):

$$\varphi(W) = \frac{(-1)^{m(n-2m)}2^{4m(n-m)}}{\pi^{2m(n-2m)}} \int_{\overline{W}' \cdot M=0} \det \|\partial_s \partial_s\|^{n-2m} \left\{ \int_{\overline{Z}' \cdot M=0} \varphi(Z)\, \omega_M \right\} \omega_W, \tag{12}$$

where $S = \overline{W}' \cdot M$, and the forms ω_M, ω_W are defined analogously to (2).

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Note: Figure translations are in progress. See original paper for figures.