Classification of the singular points of linear systems of total differential equations

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Abstract

A topological classification of the singular points of the system

$$dx^i = \sum_{\mu=1}^{n-1} \sum_{j=1}^{n} a^i_{j\mu} x^j \, dt^\mu \quad (i = 1, \dots, n),$$

is provided, assuming that the conditions of complete integrability are satisfied, and the $n \times n$-matrices $(a^i_{j\mu})$, $\mu = 1, \dots, n-1$ have no more than one pair of complex conjugate eigenvalues and no multiple eigenvalues. For $n = 3$, a complete topological classification of isolated singular points is carried out. 1 illustration. 2 bibliographical references.

Full Text

Preamble

This section, published in 1967 (TM III, No. 8), addresses the reduction of the system of differential equations (1) as presented in [1]. We consider the system:

$$\frac{dx_i}{dt} = \sum_{\mu=1}^{n-1} a_{i\mu} x_\mu$$

where $i = 1, \dots, n$. Let X be an $(n-1) \times (n-1)$ matrix $(x_{i\mu})$. We assume the rank of the matrix is $n-1$. The system (1) can be transformed into a canonical form through a change of variables.

1. Transformation of the System

Let the initial conditions be defined at $t = 0$. We seek a transformation $G \to \bar{G}$ such that the system (1) is simplified. If there exists a point x_0 in the domain G, we define the mapping $\Phi(G)$ such that the trajectories are preserved. Using the notation from [2], we introduce the complex variables $\xi = \xi_1 + i\xi_2$ and the corresponding matrix transformations.

For the $(n-1) \times (n-1)$ matrix $(x_{i\mu})$, we define the transformation:

$$x_i = \sum_{j=1}^{n-1} \beta_{ij} y_j$$

This leads to the transformed system:

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} K_{ij} y_j$$

where $i = 1, \dots, n$. The coefficients K_{ij} are determined by the eigenvalues of the original system matrix.

2. Analysis of Specific Cases

We consider the case where the coefficients $a_i \neq 0$ for $i = 1, \dots, n$. We define the auxiliary variables:

$$V = \text{sign} \sum |y_i|^{K-1}$$

The differential forms are given by:

$$d\xi_i = \text{sign}(a_i) \sum a_{i\mu} f_\mu$$

for $i = 1, \dots, n-1$. By substituting these into the original system (1), we obtain the relations between the variables y_i and the transformed coordinates ξ_i.

In the case where $a_1 \neq 0$ and $a_2 \neq 0$, we introduce the polar coordinates:

$$r = \sqrt{(y_1)^2 + (y_2)^2}, \quad \phi = \arctan(y_1/y_2)$$

The system then reduces to a set of equations for r and ϕ, which can be integrated to find the trajectories in the phase space.

3. Singular Cases and Reductions

If certain coefficients a_i vanish, the system simplifies further. For instance, if $a_i = 0$ for $i = 1, \dots, k$ (where $k < n - 1$), the corresponding variables y_i become constants of motion or evolve linearly. We define:

$$\xi_i = y_i \quad (i = 1, \dots, k)$$

The remaining equations for $i = k + 1, \dots, n - 1$ are treated using the general method described in Section 4.

For the specific case of $n = 3$, the system of equations (18) yields the following invariants:

$$l_1 l_2 = C, \quad l_1 l_2 = CF$$

where C is an integration constant. Using the substitution $l_1 = \rho \cos \phi$ and $l_2 = \rho \sin \phi$, we derive:

$$d\rho = \text{sign}(a_1/a_2) \rho dx_1, \quad d\phi = dx_2$$

This allows for a complete geometric description of the solution curves.

4. Conclusion and Applications

The proposed method allows for the systematic reduction of high-dimensional linear differential systems. By identifying the appropriate invariants and utilizing coordinate transformations (such as polar or complex representations), the complexity of the integration process is significantly reduced. These results are consistent with the fundamental theorems of differential equations as established in [1] and [2].

References

1. Pini, R. *Annali di matematica*, ser. 4, t. 34, pp. 95–104, 1953.
2. Lyapunov, A. M. *The General Problem of the Stability of Motion.* Moscow-Leningrad, 1948. (Original work submitted Jan 28, 1966).

Note: Figure translations are in progress. See original paper for figures.