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PERMUTATION GROUPS OF REPRESENTATIONS OF THE GROUPS G_2 AND SU_3

The groups G_2 and SU_3 are rather simple, since both have the fairly low rank 2. Nevertheless, they are of great practical interest, since they play an important role in the theory of the electron shells of atoms, atomic nuclei, and elementary particles. Here we shall touch on the question of permutations of the parameters of representations that leave invariant the characters of these groups.

Permutations of this kind have been well studied in the case of the group SU_2 ([1]), where they are trivial as a consequence of the fact that its rank is one. One case of a permutation for SU_3 is given in ([2]). Here we shall show that such permutations form a group characteristic of the given Lie group and isomorphic to the reflection group ([3]).

Recall that representations of the groups G_2 and SU_3 are characterized by pairs of parameters (uv) and $(\lambda\mu)$, respectively, where $u+v$ and $\lambda+\mu$ denote the number of boxes in the first row of the Young diagram, and v and μ in the second. We shall find those permutations of these parameters which leave invariant the characters of the representations, or, equivalently, the eigenvalues of the Casimir operators.

The characters of the groups considered by us are given in ([4]). They are invariant with respect to the following permutations:

Group G_2:

$$
\begin{aligned}
(uv) &\to (uv), && \text{(1a)}\\
&\to (-u-2,\ u+v+1), && \text{(1)}\\
&\to (-u-3v-5,\ u+2v+2), && \text{(1)}\\
&\to (-2u-3v-6,\ u+2v+2), && \text{(1)}\\
&\to (-2u-3v-6,\ u+v+1), && \text{(1)}\\
&\to (-u-3v-5,\ v), && \text{(1)}\\
&\to (-u-2,\ -v-2), && \text{(1)}\\
&\to (u,\ -u-v-3), && \text{(1)}\\
&\to (u+3v+3,\ -u-2v-4), && \text{(1)}\\
&\to (2u+3v+4,\ -u-2v-4), && \text{(1)}\\
&\to (2u+3v+4,\ -u-v-3), && \text{(1)}\\
&\to (u+3v+3,\ -v-2). && \text{(1)}
\end{aligned}
$$

Group SU_3:

$$
\begin{aligned}
(\lambda\mu) &\to (\lambda\mu), && \text{(2a)}\\
&\to (\mu,\ -\lambda-\mu-3), && \text{(2)}\\
&\to (-\lambda-\mu-3,\ \lambda), && \text{(2)}\\
&\to (-\mu-2,\ -\lambda-2), && \text{(2)}\\
&\to (\lambda+\mu+1,\ -\mu-2), && \text{(2)}\\
&\to (-\lambda-2,\ \lambda+\mu+1). && \text{(2)}
\end{aligned}
$$

With the aid of these permutations, from one point of the "belt figure" ([4]) one obtains all 12 or 6 points, respectively. The transformations (1) and (2) are reflections in the "belt" plane with respect to straight lines passing through the origin and forming equal-

angles with one another, and some of these elements represent the product of two reflections.

What has been said shows that the transformations given constitute a group, which it is expedient to call the transformation group of the parameters of irreducible representations.

It is not difficult to verify that group (1) is isomorphic to the point group C_{6v}, while (2) is isomorphic to the permutation group P_3. Thus they are isomorphic to the reflection groups $S(G_2)$ and $S(SU_3)$, respectively. C_{6v} has a subgroup isomorphic to the group P_3. This indicates that the group G_2, as a subgroup, contains the group SU_3. The diagram of eigenvalues (roots) of the latter is obtained from the corresponding diagram of the group G_2 by deleting the short lines passing through the origin (cf. ([3]), p. 26).

The transformations (1) and (2) also make it possible to obtain all equivalent eigenvalues from the dominant eigenvalues. However, in the present case the numerical terms in (1) and (2) must be omitted. For example, instead of (2) one should take $(\lambda\mu) \to (\mu, -\lambda-\mu)$. The fundamental dominant eigenvalues of the groups under consideration have, respectively, the form ([4])

$$\mathbf{M}(uv) = \frac{1}{2\sqrt{3}}\, u\,(1,0) + \frac{1}{4\sqrt{3}}\, v\,(3,\sqrt{3}), \tag{3a}$$

$$\mathbf{M}(\lambda\mu) = \frac{1}{6}\,\lambda\,(\sqrt{3},1) + \frac{1}{6}\,\mu\,(\sqrt{3},-1). \tag{3 }$$

From them, by the indicated method, we obtain all eigenvalues equivalent to them. In the usual way, the remaining dominant eigenvalues are obtained from (3), and from these, again in the same way, the eigenvalues equivalent to them.

It is easy to generalize the contragredience relation for the basic functions of a representation of the group SU_2, given by formula (20.2) ([1]), to the case of the groups G_2 and SU_3. They take the form

$$|(uv)t(\lambda\mu)II_{zY}\rangle^* = \left|(\bar{u}\bar{v})\bar{t}(\bar{\lambda}\bar{\mu})\bar{I}\bar{I}_z\bar{Y}\right\rangle, \tag{4a}$$

$$|(\lambda\mu)II_{zY}\rangle^* = \left|(\bar{\lambda}\bar{\mu})\bar{I}\bar{I}_z\bar{Y}\right\rangle. \tag{4 }$$

Here

$$(\bar{u}\bar{v}) = (-u-2, -v-2), \tag{5a}$$

$$(\bar{\lambda}\bar{\mu}) = (-\lambda-2, -\mu-2), \tag{5 }$$

$$\bar{I} = -I-1, \tag{5 }$$

$$\bar{I}_z = -I_z, \quad \bar{Y} = -Y; \tag{5 }$$

t denotes an additional parameter characterizing the representation of the group SU_3 as a subgroup of G_2, for which the corresponding transformation can be found. I, I_z are the parameters of the subgroup SU_2 (isospin and its projection), and Y is the hypercharge. Using (4), it is easy to write the corresponding contragredience relation for matrix elements.

As for the geometric interpretation of the transformations (5), it is a generalization of the corresponding interpretation of the group SU_2, set forth in Sections 10 and 20 of work ([1]). (5*a*)—(5) may be compared with inversion of two coordinate axes, and (5) with inversion of the two-dimensional subspace of commuting operators. Such an inversion corresponds to a generalization of mirror reflection of the subspace of commuting operators with respect to the hyperplane of the remaining dimensions of the given group. Thus the concept of mirror reflection, introduced in the representation theory of the group SU_2, can also be retained in the case of other Lie groups with rank exceeding one.

Transformation groups of representations of nontrivial Lie groups are very useful in practical respects. Their application in the indicated case

more effective than for the trivial group SU_2. This follows from the fact that the number of parameters subjected to substitutions is much larger in them, and that the order of the substitution group grows rapidly with increasing rank of the group itself.

Among the quantities of representations of Lie groups, the Clebsch–Gordan coefficients of the group SU_3 have been studied most extensively. The algebraic tables given for them in ([5,6]) can be shortened by approximately a factor of 6 if (2), (5b), and (5g) are taken into account. On the other hand, when unabridged tables are available, the use of the indicated substitutions makes it possible to check their correctness, since in this case some expressions pass into others or into themselves. In this way we checked the tables in ([5,6]) and found two misprints in the first work and four in the second.

It is obvious that groups of substitutions of representation parameters can be found for any Lie group, provided only that explicit expressions are available for the characters of the representation or for the eigenvalues of the Casimir operators. However, their practical usefulness becomes fully apparent only in concrete calculations of representation quantities. The most important of these quantities are the Clebsch–Gordan coefficients, as well as the transformation matrices for the method of reducing the direct product of several irreducible representations.

The indicated relations and the geometric interpretation of contragredience may prove useful in interpreting the relation between particles and antiparticles, since the concept of invariance may acquire a partly different meaning. The expected difference may consist in the fact that contragredience, which also includes the concept of time reversal, is interpreted as inversion of axes corresponding to commuting generating operators and inversion of the same subspace. It is only of this subspace that we can speak definitively in quantum mechanics, since the components relative to the other part of the space are not characteristic of the corresponding systems.

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Note: Figure translations are in progress. See original paper for figures.