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Abstract

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THE EXCEPTIONAL CASE OF A PAIR OF INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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It is known that the condition of normality* of the paired equation

$$\Pi f \equiv \begin{cases} \lambda_1 f(x) + \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} k_1(x-t) f(t)\, dt, & x > 0, \\ \lambda_2 f(x) + \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} k_2(x-t) f(t)\, dt, & x < 0 \end{cases} = g(x) \qquad (1)$$

in a broad class of spaces (see (^{1},^{2})), is the requirement

$$S(x, j) = \lambda_j + K_j(x) \neq 0 \quad (j = 1, 2), \qquad (2)$$

where $K_j(x) = V k_j$ is the Fourier integral with density $k_j(t)$; x is an arbitrary point of the real axis.

In the exceptional case, when zeros of integer order at individual points of the real axis are admitted for the symbol (2), equation (1) and its adjoint were studied in papers (^{4},^{5}).

Under certain restrictions on the smoothness of the coefficients $K_j(x)$, the authors of the cited papers indicated certain sufficient conditions that the free term $g(x)$ must satisfy in order that equation (1) be solvable in L_2 or in a class close to it.

Definition 1. Let $n \geq 0$ be an integer; then denote by

$$E\{\pm n; 0\} = E\{\pm n\}$$

the Banach space of functions $f(x)$, defined on the real axis, such that $(x + i)^{\mp n} f(x) \in E$, where E is one of the spaces:

$$L_p \ (p \geq 1), \quad C^0 \subset C \subset M^u \subset M^c \subset M.$$

Here the norm in the space $E\{\pm n\}$ is introduced as follows (see (^{1}), § 6):

$$\|f\|_{E\{\pm n\}} = \|(x + i)^{\mp n} f(x)\|_E.$$

By

$$E\{-\infty; 0\} = E\{-\infty\}$$

we denote the countably normed space

$$E\{-\infty\} = \bigcap_{k=1}^{\infty} E\{-k\}$$

with a set of pairwise comparable and consistent norms

$$\|f\|_k = \|f\|_{E\{-k\}} \quad (k = 1, 2, \ldots),$$

and by

$$E\{\infty; 0\} = E\{\infty\}$$

we shall mean the space

$$E\{\infty\} = \bigcup_{k=1}^{\infty} E\{k\}$$

with weak convergence.**

Each of the spaces $E\{\pm n\}$, $E\{\pm\infty\}$ decomposes into the direct sum of subspaces

$$E_{\oplus\ominus}\{\pm n\}, \quad E_{\oplus\ominus}\{\pm\infty\}$$

of functions equal to zero respectively for $x < 0$ and $x > 0$. The operator Π and its transposed operator Π^t are studied in the introduced spaces under the assumption that

$$k_j(t) \in L_1\{-n\} \quad \text{or} \quad L_1\{-\infty\}.$$

For the symbol of the operator Π,

$$S(x, j) = \lambda_j + K_j(x), \quad j = 1, 2,$$

the singularities indicated earlier are admitted. Necessary and sufficient conditions for solvability of equation (1) in the spaces

$$E\{-n\}, \quad E\{-\infty\}$$

are given. In addition, it follows from the results stated below that in the spaces

$$L_p\{-\infty\} \quad (p \geq 1)$$

Π is normally solvable and has a finite d-characteristic.

If the symbol $S(x, j)$ of the operator Π satisfies conditions (2), then for the operator Π in the spaces $E\{\pm n\}$, $E\{\pm\infty\}$ the theory of a particular

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* By a normal operator is meant an operator satisfying the Noether theory (see ([3]), p. 222).

** By $E\{\infty\}$ one should understand, generally speaking, only the spaces $L_p\{\infty\}$ $(p > 1)$, $M\{\infty\}$, each of which is conjugate to some $E\{-\infty\}$.

case $n = 0$ (see (2)). The corresponding results can be obtained by the methods of ([1],[2]).

In the exceptional case studied here, the method for investigating equation (1) consists in constructing an auxiliary operator N, by whose action on the operator Π the latter is reduced to the normal case.

Definition 2. Let the operator T be a linear bounded operator acting from a Banach space B into itself. Suppose there exists a homogeneous and additive operator N such that the operator $\widetilde{T}$, which is a composition (left or right) of the operators N and T, is normal in the space B. Then we shall say that the operator T has a **normalizer** (left or right) N in the space B. If, moreover, the sets of zeros in B of the operators T and $\widetilde{T}$ coincide, then we shall say that the operator T has an **equivalent normalizer** (left or right) N in the space B.

Without loss of generality, we shall assume that the kernels $K_j(t)$ admit the representations

$$\lambda_1+K_1(x) = \frac{A(x)}{(x-i)^{\alpha}}(\lambda_1+\widetilde{K}_1(x)), \qquad \lambda_2+K_2(x) = \frac{B(x)}{(x+i)^{\beta}}(\lambda_2+\widetilde{K}_2(x)), \quad (3)$$

where

$$A(x) = \prod_{k=1}^{r}(x-a_k)^{\alpha_k}, \qquad B(x) = \prod_{j=1}^{s}(x-b_j)^{\beta_j},$$

$$\operatorname{Im} a_k = \operatorname{Im} b_j = 0; \quad \alpha_k,\ \beta_j \text{ are positive integers}, \quad \sum_{k=1}^{r}\alpha_k = \alpha,$$

$$\sum_{j=1}^{s}\beta_j = \beta; \qquad \lambda_j + \widetilde{K}_j(x) = \lambda_j + V\tilde{k}_j, \quad j = 1, 2,$$

where $\tilde{k}_j(t) \in L_1\{-n\}$ or $L_1\{-\infty\}$, and $\lambda_j + \widetilde{K}_j(x) \neq 0,\ j = 1, 2$, on the closed real axis.

Let $f(t) \in E\{\pm n\}$ $(E\{\pm\Pi\})$. The integrals on the left-hand sides of the equalities (1), understood in the usual sense

$$\left(\int_{-\infty}^{\infty} = \lim_{M,M_1\to\infty}\int_{-M}^{M_1}\right),$$

exist almost everywhere and are bounded operators in $E\{\pm n\}, E\{\pm\infty\}$.

Theorem 1. *Let $k_j(t) \in L_1\{-n\}$ $(n > 0)$ or $L_1\{-\infty\}$, $j = 1, 2$, and let conditions (3) be satisfied. Then in the space $E\{-n\}$ $(E\{-\infty\})$ the operator Π has a left equivalent normalizer.*

Proof. The desired operator N has the form

$$N\varphi = \begin{cases} \prod\limits_{k=1}^{r}(N_{k+})^{\alpha_k}\varphi, & x > 0, \\ \prod\limits_{j=1}^{s}(N_{j-})^{\beta_j}\varphi, & x < 0, \end{cases} \tag{4}$$

where

$$N_{k+}\varphi = \varphi(x) + i(a_k - i)e^{-ia_k x}\int_x^{\infty} e^{ia_k t}\varphi(t)\,dt, \tag{5}$$

$$N_{j-}\varphi = \varphi(x) - i(b_j + i)e^{-ib_j x}\int_{-\infty}^{x} e^{ib_j t}\varphi(t)\,dt.$$

Its normalization properties are verified directly. The equivalence of the normalization follows from the following fact: the only zero of the operator N in the spaces $E\{-n\}, E\{-\infty\}$ is the function $\varphi = 0$.

From Theorem 1 it follows directly that

Corollary 1. *The number of linearly independent solutions in the spaces $E\{-n\}$, $E\{-\infty\}$ of the homogeneous equation (1) is finite and is equal to the index*

$$\tilde{\varkappa} = \operatorname{Ind}[\lambda_2 + \tilde{K}_2(x)/\lambda_1 + \tilde{K}_1(x)]$$

of the normalized operator $\tilde{\Pi} = N\Pi$, if $\tilde{\varkappa} > 0$. For $\tilde{\varkappa} \le 0$ the homogeneous equation (1) has no solutions in $E\{-n\}$, $E\{-\infty\}$ distinct from the trivial solution $f \equiv 0$.

The second corollary of Theorem 1 is a solvability criterion for the nonhomogeneous equation (1) in the space $E\{-n\}$. Denote by $DE\{-n\}$ the space of functions $g(x)$ satisfying the conditions: a) $g(x) \in E\{-n\}$; b) $Ng \in E\{-n\}$, where N is the operator of the form (4), (5).

Theorem 2. *In order that, under the assumptions of Theorem 1, the nonhomogeneous equation (1) have a solution $f \in E\{-n\}$, it is necessary and sufficient that the following conditions hold: a) $g \in DE\{-n\}$; b) if $\tilde{\varkappa} < 0$, then*

$$\int_{-\infty}^{\infty} (Ng)(t)\tilde{\varphi}_k(t)\,dt = 0 \qquad (k = 1, 2, \dots, |\tilde{\varkappa}|), \tag{6}$$

where $\{\tilde{\varphi}_k\}$ $(k = 1, 2, \dots, |\tilde{\varkappa}|)$ is a basis of the null subspace in $E\{\pm n\}$ of the transposed normalized operator $\tilde{\Pi}^\tau$.

Remark. All the results given above, in the limiting case $n = 0$, are valid for the spaces L_p $(p \geq 1)$ and C^0.

In view of the fact that the operator N is bounded in the space $E\{-\infty\}$, the analogue of Theorem 2 for this space has a simpler form:

Theorem 3. *In order that, under the conditions of Theorem 1, the nonhomogeneous equation (1) have a solution $f \in E\{-\infty\}$, it is necessary and sufficient that condition b) of Theorem 2 be fulfilled for $g \in E\{-\infty\}$.*

Let us turn to the transposed equation

$$\Pi^\tau\varphi = \lambda_1 P_+\varphi(t) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_1(x-t)P_+\varphi(x)\,dx - \lambda_2 P_-\varphi(t) -$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_2(x-t)P_-\varphi(x)\,dx = \psi(t) \qquad (-\infty \leq t \leq \infty), \tag{7}$$

where $P_\pm = \frac{1}{2}[\pm 1 + \operatorname{sign} t]$.

Let the operator $(I + K)$, where

$$(I+K)\varphi = \varphi(t) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k(t-x)\varphi(x)\,dx \qquad (-\infty \leq t \leq \infty), \tag{8}$$

be generated by a kernel $k(t) \in L_{1\oplus\ominus}\{-n\}$ or $L_{1\oplus\ominus}\{-\infty\}$. Then it leaves invariant, respectively, the subspaces $E_{\oplus\ominus}\{\pm n\}$ $(E_{\oplus\ominus}\{\pm\infty\})$.

Using this, we introduce a new unknown function $\tilde{\varphi}(t) \in E\{n\}$ $(E\{\infty\})$

$$\tilde{\varphi}(t) = (I + A_+)P_+\varphi(t) - (I + B_-)P_-\varphi(t),$$

where the operators $(I+A_+)$ and $(I+B_-)$ of type (8) have, respectively, symbols $A(-x)/(-x-i)^\alpha$ and $B(-x)/(-x+i)^\beta$. With respect to the new unknown function $\tilde{\varphi}(t)$, equation (7) takes the form

$$\tilde{\Pi}^\tau\tilde{\varphi} = \lambda_1 P_+\tilde{\varphi}(t) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{k}_1(x-t)P_+\tilde{\varphi}(x)\,dx - \lambda_2 P_-\tilde{\varphi}(t) -$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{k}_2(x-t)P_-\tilde{\varphi}(x)\,dx = \psi(t) \qquad (-\infty \leq t \leq \infty). \tag{9}$$

The operator Π^τ, by virtue of conditions (3), is normal in any of the spaces $E\{n\}$, $E\{\infty\}$, and has index $\tilde{\varkappa}_\tau = -\varkappa$. For the normal equation (9) the corresponding results can be formulated and the general solution written out. The further study of equation (7) reduces to finding the conditions for invertibility of the operators $(I + A_+)$, $(I + B_-)$ in the subspaces

$$E_{\oplus\ominus}\{n\}(E_{\oplus\ominus}\{\infty\}).$$

In the subspaces $E_{\oplus\ominus}\{\infty\}$ the indicated operators have bounded inverses $(I + A_+)^{-1}$, $(I + B_-)^{-1}$ of the form

$$(I + A_+)^{-1} = P_+ \prod_{k=1}^{r} (N_{k+}^\tau)^{\alpha_k}, \qquad (I + B_-)^{-1} = P_- \prod_{j=1}^{s} (N_{j-}^\tau)^{\beta_j},$$

where

$$N_{k+}^\tau \varphi = \varphi(t) + i(a_k - i)e^{ia_k t} \int_0^t e^{-ia_k x} \varphi(x)\, dx, \qquad t > 0,$$

$$N_{j-}^\tau \varphi = \varphi(t) - i(b_j + i)e^{ib_j t} \int_t^0 e^{-ib_j x} \varphi(x)\, dx, \qquad t < 0.$$

Hence it follows that

Theorem 4. Let $k_j(t) \in L_1\{-\infty\}$ and let conditions (3) be satisfied. Then, if $\tilde{\varkappa}_\tau > 0$, in any space $E\{\infty\}$ the homogeneous equation (7) has $\tilde{\varkappa}_\tau$ linearly independent solutions. For $\tilde{\varkappa}_\tau \leq 0$, in none of the spaces $E\{\infty\}$ does it have solutions different from the trivial one.

The nonhomogeneous equation (7), in the case $\tilde{\varkappa}_\tau \geq 0$, has a solution $\varphi \in E\{\infty\}$ for every right-hand side $\psi \in E\{\infty\}$. If, however, $\tilde{\varkappa}_\tau < 0$, then the necessary and sufficient conditions for its solvability in $E\{\infty\}$ have the form

$$\int_{-\infty}^{\infty} \psi(t)\, \tilde{f}_k(t)\, dt = 0 \qquad (k = 1, 2, \dots, |\tilde{\varkappa}_\tau|),$$

where $\{\tilde{f}_k\}$ $(k = 1, 2, \dots, |\tilde{\varkappa}_\tau|)$ is a basis of the null subspace in $E\{-\infty\}$ of the normalized operator $\tilde{\Pi}^*$.

In all spaces $E\{\infty\}$ the solutions of the homogeneous equation (7) are the same, and the solution of the nonhomogeneous equation is determined by one and the same resolvent. Therefore the form and some properties of these solutions may be found in [6], § 4, where the study of equation (7) in the space $L_1\{\infty; 0\}$ was carried out by another method. In particular, from the results of that work it follows that conditions (6) are the orthogonality conditions for the right-hand side $g \in L_p\{-\infty\}$ $(p \geq 1)$ of equation (1) to one of the possible bases of the null subspace in $L_q\{\infty\}$ $(q = p/(p-1))$, $M\{\infty\}$ of the transposed operator Π^τ.

Thus we obtain that, under assumptions (3) $(k_j \in L_1\{-\infty\}, j = 1, 2)$, the operator Π is a normally solvable operator in $L_p\{-\infty\}$ $(p \geq 1)$ with finite d-characteristic $(\tilde{\varkappa}, 0)$ or $(0, -\tilde{\varkappa})$ and index $\varkappa$. The operator Π^τ in the space $E\{\infty\}$ possesses analogous properties.

It can be shown that the operator Π, defined in the space $L_p\{-n\}$ $(p \geq 1)$, with range in the space $DL_p\{-n\}$, is also non-Noetherian.

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REFERENCES

1. M. G. Krein, *Uspekhi Mat. Nauk*, **13**, no. 5 (83) (1958).
2. I. Ts. Gokhberg, M. G. Krein, *Journal of Applied Mathematics and Mechanics*, no. 1, L' vov, 1958.
3. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.
4. F. D. Gakhov, V. I. Smagina, *Izv. Acad. Sci. USSR, Ser. Math.*, **26**, no. 3 (1962).
5. S. G. Samko, *Differential Equations*, **1**, no. 8 (1965).
6. V. B. Dybin, N. K. Karapetyants, *Siberian Mathematical Journal*, **7**, no. 3 (1966).

* And consequently also of the operator Π (Corollary 1 of Theorem 1).

Note: Figure translations are in progress. See original paper for figures.