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V. V. VELICHENKO

OPTIMALITY CONDITIONS IN CONTROL PROBLEMS WITH INTERMEDIATE CONDITIONS

(Presented by Academician L. S. Pontryagin, 5 VIII 1966)

The results of L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, generalized by the authors in the monograph (1), make it possible to formulate optimality conditions for a broad class of problems in the theory of optimal processes. In the present paper, which to a considerable extent uses the methods set forth in (1), and some results of the work (2), optimality conditions are obtained for the following problem.

1°. Statement of the problem. Consider the n-dimensional system of equations

$$\dot{x}(t) = f(x, u, t), \tag{1}$$

whose trajectory must satisfy the endpoint conditions

$$A_\alpha[x(T_0), T_0] = 0, \qquad \alpha = 0, 1, \dots, a \leq n, \tag{2}$$

$$B_\beta[x(T), T] = 0, \qquad \beta = 0, 1, \dots, b < n, \tag{3}$$

and, for $t = T_1,\ T_0 < T_1 < T$, the intermediate conditions

$$C_\gamma[x(T_1), T_1] = 0, \qquad \gamma = 0, 1, \dots, c < n. \tag{4}$$

In what follows, assuming that

$$dA_0(x,t)/dt\big|_{t=T_0} \neq 0, \qquad dB_0(x,t)/dt\big|_{t=T} \neq 0, \qquad dC_0(x,t)/dt\big|_{t=T_1-0} \neq 0,$$

we shall use the conditions

$$A_0[x(T_0), T_0] = 0, \qquad B_0[x(T), T] = 0, \qquad C_0[x(T_1), T_1] = 0 \tag{5}$$

to determine the instants of time T_0, T, and T_1.

As the class of admissible controls we take r-dimensional piecewise-continuous vector functions $u(t)$, whose values must belong to a prescribed closed domain U.

Let the functional

$$I(x, u) = \Phi[x(T), T]. \tag{6}$$

We pose the following problem. **Among all controls $u(t) \in U$ such that the trajectories of system (1) corresponding to these controls satisfy conditions (2), (3), and (4), choose one for which the functional (6) takes the minimal value.**

The functions $A_\alpha(x,t)$, $B_\beta(x,t)$, $C_\gamma(x,t)$, and $\Phi(x,t)$ are assumed to be continuous together with their first-order partial derivatives and to have bounded second-order partial derivatives with respect to the arguments x, t. The vector function $f(x,u,t)$ will be assumed to have the indicated properties with respect to the aggregate of the arguments x, u and to be continuous in t.

2°. Optimality conditions. Introduce the notation

$$A(x,t) = \sum_{\alpha=1}^{a} \lambda_\alpha^A A_\alpha(x,t), \qquad B(x,t) = \sum_{\beta=1}^{b} \lambda_\beta^B B_\beta(x,t) + \lambda^\Phi \Phi(x,t),$$

$$C(x,t) = \sum_{\gamma=1}^{c} \lambda_\gamma^C C_\gamma(x,t).$$

Theorem 1 (maximum principle). If the control $u(t)$ and the trajectory $x(t)$ are optimal in problem (1)–(6), then there exist numbers λ_α^A, λ_β^B, λ_γ^C, and λ^Φ satisfying the conditions

$$\sum_{\alpha=1}^{a} (\lambda_\alpha^A)^2 + \sum_{\beta=1}^{b} (\lambda_\beta^B)^2 + \sum_{\gamma=1}^{c} (\lambda_\gamma^C)^2 + (\lambda^\Phi)^2 = 1, \qquad \lambda^\Phi \geqslant 0,$$

and such a vector-function $p(t)$, satisfying the system of equations ([1])

$$\dot{p}(t) = -\operatorname{grad}_x H(x, p, u, t), \qquad T_0 \leqslant t < T_1, \quad T_1 < t \leqslant T, \tag{7}$$

where $H(x, p, u, t) \equiv (p, f(x, u, t))$, the boundary conditions

$$p(T_0) = \left[\operatorname{grad}_x A(x,t) - \left(\frac{dA(x,t)}{dt} \Big/ \frac{dA_0(x,t)}{dt}\right) \operatorname{grad}_x A_0(x,t)\right]_{t=T_0}, \quad (8)$$

$$p(T) = \left[-\operatorname{grad}_x B(x,t) + \left(\frac{dB(x,t)}{dt} \Big/ \frac{dB_0(x,t)}{dt}\right) \operatorname{grad}_x B_0(x,t)\right]_{t=T} \quad (9)$$

and the jump condition

$$p(T_1 - 0) - p(T_1 + 0) \quad (10)$$

$$= \left[-\operatorname{grad}_x C(x,t) + \left(\left[\frac{dC(x,t)}{dt} - \mu\right] \Big/ \frac{dC_0(x,t)}{dt}\right) \operatorname{grad}_x C_0(x,t)\right]_{t=T_1-0},$$

where

$$\mu = (p(T_1 + 0), [f[x(T_1), u(T_1 - 0), T_1] - f[x(T_1), u(T_1 + 0), T_1]]),$$

such that the maximum condition is fulfilled

$$H(x, p, u, t) = \sup_{v \in U} H(x, p, v, t), \qquad T_0 < t < T_1, \quad T_1 < t < T. \quad (11)$$

The method of proving Theorem 1 and the assertions formulated below is based on studying cones of attainability ([1]) in an $(a + b + c + 1)$-dimensional vector space, along whose axes are laid off variations of the functional (6) and of the left-hand sides of the equalities (3), (4), and (2) for $\beta = 1, \dots, b$, $\gamma = 1, \dots, c$, and $\alpha = 1, \dots, a$, calculated under the condition that, for the varied trajectory ([1]) of system (1), the second, third, and first equalities (5), respectively, are satisfied. The indicated variations can be written in a form analogous to that used in ([2]), with a remainder term whose estimate, in the case of a fixed left endpoint of the trajectory, coincides with that obtained in ([2]).

Theorem 2. For optimality of the control $u(t)$ "in the small" ([2]) on an interval $[\tau_1, \tau_2]$, containing none of the time instants T_1 and T, in problem (1)–(6) with fixed left endpoint of the trajectory, it is sufficient that there exist numbers λ_β^B, λ_γ^C, and λ^Φ such that $\lambda^\Phi > 0$ and the function $H(x, p, u, t)$, determined by conditions (7), (9), and (10), satisfies the maximum condition (11) on the interval $[T_0, T]$ and the conditions of Theorem 1 of the work ([2]) on the interval $[\tau_1, \tau_2]$.

3°. Linear systems. Let system (1) be linear in x,

$$\dot{x}(t) = F(t)x + \varphi(t, u), \tag{12}$$

the time instants T_0, T_1, and T be fixed and the conditions (2)–(4) and the function-

were given in (6) in the form

$$\begin{aligned} A_\alpha[x(T_0), T_0] &\equiv (l^A_\alpha, x(T_0)) + m^A_\alpha = 0, \quad \alpha = 1, \dots, a, \\ B_\beta[x(T), T] &\equiv (l^B_\beta, x(T)) + m^B_\beta = 0, \quad \beta = 1, \dots, b, \\ C_\gamma[x(T_1), T_1] &\equiv (l^C_\gamma, x(T_1)) + m^C_\gamma = 0, \quad \gamma = 1, \dots, c, \\ \Phi[x(T), T] &\equiv (l^\Phi, x(T)), \end{aligned} \tag{13}$$

where the vectors l and the numbers m are constants.

Theorem 3. *For the optimality of the control $u(t)$ and the trajectory $x(t)$ in problem (12), (13), it is sufficient that there exist numbers λ^A_α, λ^B_β, λ^C_γ, and λ^Φ, $\lambda^\Phi > 0$, and such a vector-function $p(t)$, satisfying conditions (7)–(10), that the maximum condition (11) is fulfilled.*

Let $X(t)$ be a fundamental matrix of the system of solutions of the homogeneous system of equations corresponding to system (12). We shall call problem (12), (13), for which $a + b + c = n$ and the vectors $X'(T_0)l^A_\alpha$, $\alpha = 1, \dots, a$, $X'(T_1)l^C_\gamma$, $\gamma = 1, \dots, c$, and $X'(T)l^B_\beta$, $\beta = 1, \dots, b$, are linearly independent, **simplest**.

Theorem 4. *For simplest problems, the conditions of Theorem 3 are sufficient and necessary.*

The results obtained are easily generalized to the case where there are several systems of intermediate conditions of the form (4), and also to the case where $T_1 > T$.

4°. Equations with discontinuous right-hand sides. The following problem (1) reduces to the formulation considered. Suppose that system (1) has a discontinuity at the time $t = T_1$, determined by the fulfillment of the condition $C_0[x(T_1), T_1] = 0$, so that

$$\dot{x}(t) = f^-(x, u, t) \quad \text{for } t < T_1, \qquad \dot{x}(t) = f^+(x, u, t) \quad \text{for } t > T_1. \tag{14}$$

Assuming that the left and right systems (14) determine the variation of different phase coordinates $x^-(t)$ and $x^+(t)$, and requiring fulfillment of the conditions

$$C_\gamma[x^-(T_1), x^+(T_1)] \equiv x^-_\gamma(T_1) - x^+_\gamma(T_1) = 0, \qquad \gamma = 1, \dots, n,$$

we arrive at a problem with intermediate conditions.

Theorem 5. *If the control $u(t)$ and the trajectory $x(t)$ of system (14) are optimal in the problem of minimizing the functional (6) under conditions (2) and (3), then there exist numbers λ_α^A, λ_β^B, and λ^Φ, satisfying the condition*

$$\sum_{\alpha=1}^{a}(\lambda_\alpha^A)^2 + \sum_{\beta=1}^{b}(\lambda_\beta^B)^2 + (\lambda^\Phi)^2 = 1, \qquad \lambda^\Phi \geq 0,$$

and such a vector-function $p(t)$, satisfying conditions (7)–(9) and the jump condition

$$p(T_1 - 0) - p(T_1 + 0) = -\nu \operatorname{grad}_x C_0[x(T_1), T_1],$$

where

$$\nu = \frac{(p(T_1+0), [\, f^-[x(T_1), u(T_1-0), T_1] - f^+[x(T_1), u(T_1+0), T_1]\,])}{(\operatorname{grad}_x C_0[x(T_1), T_1], f^-[x(T_1), u(T_1-0), T_1]) + \partial C_0[x(T_1), T_1]/\partial T_1},$$

that the maximum condition (11) is fulfilled.

Moscow Institute of Physics and Technology

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Note: Figure translations are in progress. See original paper for figures.