Contact problem for a linear deformable foundation

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Abstract

A spatial contact problem regarding the pressure of a strip punch on an elastic foundation with an elastic modulus $E = E_m z^m$ $(0 < m < 1)$ is solved. Using the method of Fourier integral transforms, the problem is reduced to dual integral equations of trigonometric type. By means of an existing solution to these equations, a new formula for the pressure under the punch is obtained, which includes certain auxiliary functions determined from integral equations of the second kind. Bibliography: 5 items.

Full Text

Introduction

In 1967, B. S. Pyatigorskii investigated the integral equation of the form:

$$w(x, y) = k \iint_S \frac{p(\xi, \eta)}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\frac{m+1}{2}}} d\xi d\eta, \quad 0 < m < 1 \tag{0.1}$$

Equations of this type arise in various problems in the theory of elasticity and potential theory, particularly when considering the pressure of a rigid punch on an elastic half-space or the distribution of charge on a conductor [1-3]. The function $w(x, y)$ is typically prescribed over a region S, and the objective is to determine the unknown density $p(x, y)$.

1. Problem Formulation and Methodology

Consider the boundary value problem for the potential $\Phi(x, y, z)$ in the half-space $z > 0$:

$$\Delta\Phi = 0 \tag{1.1}$$

subject to the boundary conditions:

$$\Phi(x, y, 0) = w(x, y) \in (S) \tag{1.2}$$

$$\lim_{z\to 0} z^m \frac{\partial \Phi}{\partial z} = 0, \quad (x,y) \notin (S)$$

where Φ vanishes at infinity such that $\Phi \sim O(r^{m-1})$ as $r = \sqrt{x^2+y^2+z^2} \to \infty$. Here, (S) represents the contact area. In previous studies [2], it was shown that for a strip-like region (S), the solution to equation (1.1) under conditions (1.2) can be sought in the form:

$$\Phi(x,y,z) = X(x,z)\cos\beta y$$

Substituting this into the Laplace equation (1.1) leads to the reduced equation:

$$\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial z^2} - \beta^2 X = 0 \tag{1.3}$$

The solution for the potential $X(x,z)$ can be expressed using the modified Bessel function of the second kind $K_\nu(r)$:

$$X(x,z) = \int_0^\infty A(\lambda) e^{-z\sqrt{\lambda^2+\beta^2}} \cos\lambda x \, d\lambda$$

Applying the boundary conditions (1.2) leads to a system of dual integral equations for the unknown function $A(\lambda)$:

$$\begin{gathered} \int_0^\infty [1-g(\lambda)]A(\lambda)\cos\lambda x\, d\lambda = f_1(x), \quad (x<a) \\ \int_0^\infty \lambda A(\lambda)\sin\lambda x \, d\lambda = 0, \quad (x>a) \end{gathered} \tag{1.4}$$

where $f_1(x)$ and $f_2(x)$ are given functions related to the geometry and loading, and $g(\lambda)$ is a kernel function that accounts for the specific boundary constraints.

2. Solution via Fredholm Integral Equations

The system of dual integral equations (1.4) can be reduced to a Fredholm integral equation of the second kind using the method described in [4]. Let the auxiliary functions $h_0(t)$ and $h_1(t)$ be defined such that:

$$h_i(x) = \int_0^a h_i(t) K_i(x,t)\, dt + \omega_i(x), \quad (i=0,1) \tag{1.8}$$

The kernel $K_i(x,t)$ is given by:

$$K_i(x,t) = \int_0^\infty g(\lambda) J_\nu(\lambda t) J_\nu(\lambda x)\, d\lambda \tag{1.9}$$

The density function $p(x)$ can then be recovered from $h_0(x)$ and $h_1(x)$ through the relation:

$$p(x) = \frac{h_0(a) + x h_1(a)}{\sqrt{a^2-x^2}} + \int_x^a \frac{h_0'(t) + x h_1'(t)}{\sqrt{t^2-x^2}} dt \tag{1.13}$$

In the specific case where $w(x,y) = d + ex$, the functions $\omega_i(x)$ simplify, allowing for a direct numerical or analytical solution of the integral equations (2.1).

3. Applications and Examples

As an application, consider the case where $w(x, y) = d$, representing a flat rectangular punch. If we assume the contact area is a strip of width $2a$, the pressure distribution $p(x)$ can be determined using the derived relations. For $m = 0$, the results coincide with the classical solutions for a rigid punch on an elastic half-space [5].

Further analysis of the case $w(x, y) = d - \beta x$ (see Fig. 1) shows that the pressure distribution remains bounded at the edges if the parameters are chosen to satisfy the equilibrium conditions. The results obtained via equation (1.13) are consistent with the asymptotic behavior expected for such mixed boundary value problems in linear elasticity.

References

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Note: Figure translations are in progress. See original paper for figures.