ON THE APPROXIMATION OF PERIODIC FUNCTIONS BY LINEAR METHODS OF SUMMATION OF FOURIER SERIES

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.512.6

MATHEMATICS

V. V. ZHUK

ON THE APPROXIMATION OF PERIODIC FUNCTIONS BY LINEAR METHODS OF SUMMATION OF FOURIER SERIES

(Presented by Academician L. V. Kantorovich on 27 IV 1966)

The work is a continuation of the paper ([1]).

1°. Notation and assumptions. A function $f(x) \in C_{2\pi}$; $\omega_k(\delta, f)$ is its modulus of smoothness of order k; $E_n(f)$ denotes best approximations by trigonometric polynomials of order $\leq n$; $T_n(x, f)$ is a polynomial of best approximation of order n; $\tilde{f}(x)$ is the function trigonometric-conjugate to $f(x)$; $T_n(x)$ is a trigonometric polynomial of order $\leq n$; the numbers r, k, n are natural; $U(f)$ is a bounded subadditive (i.e. $\|U(f+g)\| \leq \|U(f)\| + \|U(g)\|$) operator from $C_{2\pi}$ to $C_{2\pi}$; $U(f) \in A_n$, if for every $T_n(x)$: 1) $U(T_n)$ is a trigonometric polynomial of order $\leq n$, 2) $U(T_n) = \widetilde{U}(T_n)$, 3) for any r, $U^{(r)}(T_n) = U(T_n^{(r)})$; $C(0 < C < \infty)$ and $M(0 \leq M < \infty)$ are constants depending only on those arguments that will be indicated.

2°. Main theorems. Let the even functions $\Phi_n^{[k]}(t) \geq 0$ $(k = 1, 2, \dots, l)$ be given on $[-\pi, \pi]$ and possess the following properties:

$$\int_{-\pi}^{\pi} \Phi_n^{[k]}(t)\, dt = 1, \qquad \Delta_n^{[k]} = \int_0^{\pi} t^2 \Phi_n^{[k]}(t)\, dt \xrightarrow[n\to\infty]{} 0 \quad (k = 1, \dots, l).$$

Put

$$U_n^{[k]}(f) = f(x) - \int_{-\pi}^{\pi} f(x+t) \Phi_n^{[k]}(t)\, dt,$$

$$\Pi_n^{[1]}(f) = U_n^{[1]}(f), \qquad \Pi_n^{[k]}(f) = U_n^{[k]}(\Pi_n^{[k-1]}(f)) \quad (k = 1, \dots, l).$$

Theorem 1. For all $n \geq 1$,

$$\|\Pi_n^{[l]}(f)\| \leq C_1(l)\,\omega_{2l}\left(\left(\prod_{k=1}^{l}\Delta_n^{[k]}\right)^{1/2l},\ f\right).$$

This theorem generalizes some results of I. P. Natanson ([2]).

Theorem 2. Let $\Phi_n^{[k]}(t)$ be the classical Jackson kernel (see ([3]), p. 115) ($k = 1, 2, \ldots, l$). Then for $n \geq 2$,

$$\omega_{2l}\left(\frac{1}{n},\ f\right) \leq C_2(l)\,[E_{n-2}(f) + \|\Pi_n^{[l]}(f)\|].$$

Corollary 1. Under the conditions of Theorem 2,

$$\omega_{2l}\left(\frac{1}{n},\ f\right) \leq C_3(l)\sup_{m\geq n}\|\Pi_m^{[l]}(f)\|.$$

Remark 1. An analogous theorem is also true for the Jackson-Vallee-Poussin kernel.

Theorem 3. If for every $T_n(x)$

$$\big|\|U(T_n)\| - M_1\|T_n^{(k)}(x)\|/n^k\big| \leq C_4\|T_n^{(k+1)}(x)\|/n^{k+1},$$

then for any $f(x)$

$$\|U(f)\| = M_1\omega_k(1/n, f) + O\big[(\|U\| + C_5(k))\omega_{k+1}(1/n, f)\big],$$

where O depends only on k, and $C_5(k)$ on k, C_4, and M.

Theorem 4. If for any $T_n(x)$

$$\big|\|U(T_n)\| - M_2\|T_n^{(k)}(x)\|/n^k\big| \leq C_6\|\widetilde{T}_n^{(k+1)}(x)\|/n^{k+1},$$

then for any $f(x)$

$$\|U(f)\| = M_2\omega_k\left(\frac{1}{n}, f\right) + O\left[(\|U\| + C_7(k))n^{-(k+1)}\sum_{l=0}^{n-1}(l+1)^k E_l(f)\right],$$

where O depends only on k, and $C_7(k)$ only on k, C_6, and M_2.

Theorem 5. Let $U_k(f) \in A_n$ $(k = 1, 2)$.

1) If for any $T_n(x)$

$$\big|\|U_1(T_n)\| - M_3\|T_n^{(r)}(x)\|\big| \le C_8\|T_n^{(r+1)}(x)\|,$$

$$\big|\|U_2(T_n)\| - M_4\|T_n^{(k)}(x)\|\big| \le C_9\|T_n^{(k+1)}(x)\|,$$

then for any $T_n(x)$

$$\big|\|U_1[U_2(T_n)]\| - M_3M_4\|T_n^{(r+k)}(x)\|\big| \le [M_3C_9 + M_4C_8 + nC_8C_9]\|T_n^{(r+k+1)}(x)\|.$$

2) If for any $T_n(x)$

$$\big|\|U_1(T_n)\| - M_5\|T_n^{(r)}(x)\|\big| \le C_{10}\|\widetilde{T}_n^{(r+1)}(x)\|,$$

$$\big|\|U_2(T_n)\| - M_6\|T_n^{(k)}(x)\|\big| \le C_{11}\|\widetilde{T}_n^{(k+1)}(x)\|,$$

then for any $T_n(x)$

$$\begin{aligned}&\big|\|U_1[U_2(T_n)]\| - M_5M_6\|T_n^{(r+k)}(x)\|\big| \\ &\qquad \le [M_5C_{11} + M_6C_{10} + nC_{10}C_{11}]\|\widetilde{T}_n^{(r+k+1)}(x)\|.\end{aligned}$$

Theorem 6. Let $U_k(f) \in A_n$ $(k = 1, 2)$, A_k $(k = 1, 2, 3, 4)$ be real numbers.

1) If for any $T_n(x)$

$$\left\|U_1(T_n) - T_n(x) + \frac{A_1}{n^r}T_n^{(r)}(x)\right\| \le \frac{C_{12}}{n^{r+1}}\|T_n^{(r+1)}(x)\|,$$

$$\left\|U_2(T_n) - T_n(x) + \frac{A_2}{n^k}T_n^{(k)}(x)\right\| \le \frac{C_{13}}{n^{k+1}}\|T_n^{(k+1)}(x)\|,$$

then for any $T_n(x)$

$$\left\|U_1(U_2(T_n)) - T_n(x) + \frac{A_1}{n^r}T_n^{(r)}(x) + \frac{A_2}{n^k}T_n^{(k)}(x)\right\| \le \frac{C_{14}}{n^{\gamma}}\|T_n^{(\gamma)}(x)\|,$$

where C_{14} depends only on A_1, A_2, C_{12}, C_{13}, and $\gamma = \min(r + 1, k + 1)$.

2) If for any $T_n(x)$

$$\left\|U_1(T_n) - T_n(x) + \frac{A_3}{n^r} T_n^{(r)}(x)\right\| \leq \frac{C_{15}}{n^{r+1}} \|\widetilde{T}_n^{(r+1)}(x)\|,$$

$$\left\|U_2(T_n) - T_n(x) + \frac{A_4}{n^k} T_n^{(k)}(x)\right\| \leq \frac{C_{16}}{n^{k+1}} \|\widetilde{T}_n^{(k+1)}(x)\|,$$

then for any $T_n(x)$

$$\left\|U_1(U_2(T_n)) - T_n(x) + \frac{A_3}{n^r} T_n^{(r)}(x) + \frac{A_4}{n^k} T_n^{(k)}(x)\right\| \leq \frac{C_{17}}{n^{\gamma}} \|\widetilde{T}_n^{(\gamma)}(x)\|,$$

where C_{17} depends only on A_3, A_4, C_{15}, C_{16}, and $\gamma = \min(k+1, r+1)$.

3°. Some applications of the main theorems.

Example 1. Let $U_n(x, f) = U_n(f) \in A_n$ be linear operators for which $\|U_n\| \leq C_{18}$ $(n = 1, 2, ...)$. If

$$\|U_n(f) - f(x)\| \leq C_{19}\omega_2\left(\frac{1}{n}, f\right),$$

then

$$\|U_n(x + 1/n, f) - f(x)\| = \omega_1(1/n, f) + O[\omega_2(1/n, f)].$$

Proof. It is clear that for any $T_n(x)$

$$\|T_n(x + 1/n) - T_n(x) - T_n'(x)/n\| \leq \|T_n''(x)\|/2n^2.$$

By Theorem 6, for any $T_n(x)$

$$\|U_n(x + 1/n, T_n) - T_n(x) - T_n'(x)/n\| \leq C_{20}\|T_n''(x)\|/n^2$$

and, consequently,

$$|\|U_n(x + 1/n, T_n) - T_n(x)\| - \|T_n'(x)\|/n| \leq C_{20}\|T_n''(x)\|/n^2.$$

Taking Theorem 3 into account, we obtain our assertion.

Corollary. Let, under the conditions of the example, $U_n(f)$ $(n = 1, 2 \ldots)$ be a trigonometric polynomial of order not exceeding n. Then the conditions $f(x) \in \operatorname{Lip} 1$ and $\|U_n(x + 1/n, f) - f(x)\| = O(1/n)$ are equivalent.

For the proof it is enough to compare Example 1 and the theorem of A. Zygmund (see ([3]), p. 142).

Remark. Under the conditions of the corollary, the relations $f(x) \in \operatorname{Lip} 1$ and

$$\|U_n(f) - f(x) + U_n'(f)/n\| = O(1/n)$$

are equivalent.

Example 2. Let $\Phi_n(t)$ be the classical Jackson kernel,

$$I_n^{(r)}(f) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x + t_1 + \cdots + t_r)\Phi_n(t_1) \cdots \Phi_n(t_r)\, dt_1 \cdots dt_r$$

the r-th Jackson integral. Then

$$\|I_n^{(r)}(f) - f(x)\| = \frac{3r}{2}\,\omega_2\left(\frac{1}{n}, f\right) + O\left[n^{-3}\sum_{k=0}^{n-1}(k+1)^2 E_k(f)\right],$$

where O depends only on r.

Example 3. Let $\Phi_n^{[k]}(t)$ $(k = 1, 2, \dots, l)$ be the classical Jackson kernel. Then

$$\|\Pi_n^{[l]}(f)\| = \left(\frac{3}{2}\right)^l \omega_{2l}\left(\frac{1}{n}, f\right) + O\left[n^{-(2l+1)}\sum_{k=0}^{n-1}(k+1)^{2l} E(f)\right],$$

where O depends only on l.

4°. Constructive characteristics of some classes of functions.

The questions considered in the present section were also studied in the works ([4]–[8]).

Theorem 7. For every r

$$\omega(1/n, f) \le 2^{r+1}\|f(x) - T_n(x, f) + T_n^{(r)}(x, f)/n^r\| \le C_{21}(r)\omega_r(1/n, f), \qquad (1)$$

$$\omega_1(1/n, f) \le C_{22}\|f(x) - T_n(x + 1/n, f)\| \le C_{23}\omega_1(1/n, f).$$

Remark. The upper estimate in inequalities (1) is easily obtained from the results of S. B. Stechkin ([4]).

Corollary 1. In order that $f(x) \in \mathrm{Lip}\,1$, it is necessary and sufficient that

$$\|f(x) - T_n(x + 1/n, f)\| = O(1/n).$$

All the preceding results are also valid for the space $L^p_{2\pi}$ $(1 \le p < \infty)$.

Theorem 8. Let $v(x)$ be a uniformly continuous function bounded on the entire axis; let $g_\sigma(x, v)$ be an integral function of degree not exceeding σ, least deviating from $v(x)$ on $(-\infty, \infty)$ in the space $C[-\infty, \infty]$. Then

$$\omega_r(1/\sigma, v) \le C_{24}(r) \left\| v(x) - g_\sigma(x, v) + g_\sigma^{(r)}(x, v)/\sigma^r \right\| \le C_{25}(r)\omega_r(1/\sigma, v),$$

$$\omega_1(1/\sigma, v) \le C_{26} \|v(x) - g_\sigma(x + 1/\sigma, v)\| \le C_{27}\omega_1(1/\sigma, v).$$

The theorem is also valid for the spaces $L^n_{[-\infty,\infty]}$ $(1 \le p < \infty)$. Let $g(x) \in C[0, 1]$, $B_n(x, g)$ be its S. N. Bernstein polynomial, and let $x_n \in [0, 1]$ be points at which $|g(x) - B_n(x, g)|$ attains its maximum value on $[0, 1]$. Put

$$S_n(g) = B_n(g) - \gamma B_n'(g)/\sqrt{n}, \qquad V_n(g) = B_n(g) - \lambda B_n''(g)/n,$$

where

$$\gamma = \mathrm{sign}[g(x_n) - B_n(x_n, g)][B_n'(x_n, g)],$$

and

$$\lambda = \mathrm{sign}[g(x_n) - B_n(x_n, g)][B_n''(x_n, g)].$$

Theorem 9. In order that $g(x) \in \mathrm{Lip}\,1$ on $[0, 1]$, it is necessary and sufficient that

$$\|S_n(g) - g(x)\|_{C[0,1]} = O(1/\sqrt{n}).$$

In order that $g'(x) \in \mathrm{Lip}\,1$ on $[0, 1]$, it is necessary and sufficient that

$$\|V_n(g) - g(x)\|_{C[0,1]} = O(1/n).$$

The author expresses deep gratitude to G. I. Natanson for his attention to this work.

Leningrad Electrotechnical Institute
named after V. I. Ulyanov (Lenin)

Received
12 IV 1966

REFERENCES

1. V. A. Zhuk, DAN, 169, No. 3 (1966).
2. I. P. Natanson, DAN, 82, No. 3, 337 (1952).
3. I. P. Natanson, *Constructive Theory of Functions*, 1949.
4. S. B. Stechkin, Izv. AN SSSR, ser. matem., 15, No. 3, 219 (1951).
5. S. B. Stechkin, DAN, 83, No. 5, 651 (1952).
6. S. B. Stechkin, Izv. AN SSSR, ser. matem., 20, 197 (1956).
7. N. K. Bari, S. B. Stechkin, Tr. Mosk. matem. obshch., 5, 481 (1956).
8. R. M. Tregub, Izv. AN SSSR, ser. matem., 29, No. 3, 615 (1965).

Note: Figure translations are in progress. See original paper for figures.