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**Abstract**

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**MATHEMATICS**

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## PLANE AND AXISYMMETRIC FILTRATION PROBLEMS WITH FREE BOUNDARIES IN AN INHOMOGENEOUS MEDIUM

*(Presented by Academician P. Ya. Kochina, 25 VIII 1966)*

As is known (<sup>1-3</sup>), the piezometric head in plane filtration problems in an inhomogeneous medium satisfies the equation

$$\frac{\partial}{\partial x} \left( \hat{k} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( \hat{k} \frac{\partial h}{\partial y} \right) = 0$$

and the equation

$$\frac{\partial}{\partial x} \left( \hat{k}x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( \hat{k}x \frac{\partial h}{\partial y} \right) = 0$$

in axisymmetric problems, where  $\hat{k} = \hat{k}(x, y, h, h_x, h_y)$  is the filtration coefficient.

Introducing the stream function and the generalized potential  $\varphi = -k_0 h$  ( $k_0 = \text{const}$ ), in plane problems we arrive at the system of equations

$$\psi_y = k\varphi_x, \quad -\psi_x = k\varphi_y, \quad (1)$$

where  $k = k/k_0$  is the reduced filtration coefficient (the case  $k = 1$  corresponds to a homogeneous medium). In the axisymmetric case, in equations (1) it is necessary to replace  $k$  by  $kx$ . The boundary conditions in filtration problems in an inhomogeneous medium obviously coincide with the corresponding conditions of filtration problems in homogeneous media, with the role of the filtration coefficient in these conditions being played by the constant  $k_0$ .

In the present paper we shall restrict ourselves to considering those classes of filtration problems with free boundaries in which the image  $D_w$  ( $w = \varphi + i\psi$ ) of the sought filtration domain  $D_z$  ( $z = x + iy$ ) is known (in the axisymmetric case it is assumed that in the domain  $D_z$ ,  $x \neq 0$ ). This occurs in most filtration

problems even in the presence of seepage intervals on the boundary  $L_z$  of the domain  $D_z$ , taking into account the remark in <sup>(4)</sup> on the actual equivalence of the condition  $\varphi + k_0 y = \varphi_0$ , usually used on this interval, and the condition  $\psi = \text{const}$ .

Thus the original filtration problems are reduced to the following problem:

Find a pair of functions  $x = x(\varphi, \psi)$ ,  $y = y(\varphi, \psi)$ , satisfying the system of equations

$$x_\varphi = ky_\psi, \quad -y_\varphi = kx_\psi \quad (2)$$

(in the axisymmetric case  $k$  is replaced by  $kx$ ) and mapping the domain  $D_w$  onto the sought domain  $D_z$ . In this case the boundary  $L_w$  of the domain  $D_w$  is divided into parts  $L_w^1$  and  $L_w^2$ , the image  $L_z^1$  of the first of which is known, while on the second  $L_w^2$ —the image of the free boundary  $L_z^2$ —the relation  $\varphi + k_0 y = \varphi_0$  holds, i.e., on  $L_w^2$  the function  $y = y(\varphi)$  is prescribed. This so-called inverse mixed boundary-value problem, by the method of inscribed polygons and subsequent passage to the limit, was studied in a number of works by the author <sup>(5, 6)</sup> in the case of analyticity of the function  $z = z(w)$ , which corresponds to plane filtration problems in a homogeneous medium ( $k = 1$ ).

By introducing the planes of variables  $x, \psi$  or  $\varphi, y$ , with the use of the theory of quasiconformal mappings, the author <sup>(4)</sup> proved the solvability of a large class of filtration problems with free boundaries when the filtration coefficient  $\hat{k}$  was assumed to be a known function of the filtration velocity  $q = (\psi_x^2 + \psi_y^2)^{1/2}$  (the domain  $D_w$ , generally speaking, was unknown).

Let now  $0 < \hat{k}(x, y, \varphi, \psi) < \infty$  be an arbitrary function defined for  $(\varphi, \psi) \in \overline{D}_w$  and all  $(x, y)$ . Let the known portions  $L_z^1$  of the boundary of the filtration domain  $D_z$  consist of segments of straight lines  $y = \text{const}$  and curves with equations of the form  $x - f_i(y) = 0$ , where  $f_i(y)$  are single-valued piecewise differentiable functions defined for  $y \in [a_i, b_i]$  (on individual portions it is allowed that  $f_i(y) \equiv \text{const}$ ), and the intervals  $(a_i, b_i)$  do not intersect one another. It is assumed that the portions of the free boundary  $L_z^2$  (on them  $\varphi + x_0 y = \text{const}$ ) have only one common endpoint with one of the curves with equations of the form  $x - f_i(y) = 0$ , and an arbitrary finite number of common endpoints with the segments  $y = \text{const}$ ; moreover, the sought domain  $D_z$  may also be unbounded. As is easy to verify, the conditions listed above are satisfied by a large class of filtration problems. Let us name some of them <sup>(1)</sup>: problems of filtration in earth dams on an impermeable foundation ( $y = \text{const}$ ) or with backwater ( $y = \text{const}$ ) in the presence of drainage on the foundation and with an arbitrary form of the upstream face; problems on the influx of fluid to channels of arbitrary shape (the plane case) or to wells (the axisymmetric case), and, conversely, filtration from channels and wells; problems of unconfined filtration between several water-bearing strata, and others.

Introduce new unknown functions

$$t = x - f_i(y), \quad y = y, \quad y \in [a_i, b_i]$$

(in particular, it is possible that  $f_i \equiv \text{const}$ ).

Then, by virtue of the conditions on the known portions of the boundary  $L_z$ , its image in the plane  $w = \varphi + i\psi$  is split into two adjacent arcs  $l_1$  and  $l_2$  (possibly passing through the point at infinity), on one of which  $t = 0$ , while on the other  $y = \frac{1}{k_0}(\varphi_0 - \varphi)$  or  $y = \text{const}$ , i.e.  $y = y(\varphi)$  is a function prescribed on  $l_2$ . Map conformally the domain  $D_w$  onto the circle  $|\zeta| < 1$  by the function  $\zeta = F(w)$  so that the arc  $l_1$  passes into the upper half of the circumference, and transform the system of equations (2) to the new independent variables  $\xi, \eta$  ( $\zeta = \xi + i\eta$ ) and the new unknown functions  $t, y$  ( $t + iy = \omega$ ). The system (2) transformed in this way can be written in the form of one complex equation

$$\omega_{\bar{\zeta}} - q_1(\omega, \zeta)\omega_{\zeta} - q_2(\omega, \zeta)\overline{\omega_{\zeta}} = 0, \quad (3)$$

where  $q_1, q_2$  are expressed explicitly in terms of the functions  $k(x, y, \varphi, \psi)$ ,  $f_i(y)$ , and  $F(\zeta)$ , and  $|q_1| + |q_2| \leq q_0 < 1$  (the ellipticity condition of the obtained system). On the boundary of the circle  $|\zeta| = |e^{i\gamma}| = 1$  the boundary conditions

$$\text{Re } \omega = 0, \quad \gamma \in [0, \pi]; \quad \text{Im } \omega = y[\varphi(\gamma)], \quad \gamma \in [\pi, 2\pi]. \quad (4)$$

are satisfied.

We shall seek the solution of this problem in the form  $\omega = T_2 f + \overline{\Phi(\zeta)}$ , where  $\overline{\Phi(\zeta)}$  satisfies conditions (4) and is conjugate to an analytic function, while the function  $T_2 f$  satisfies homogeneous conditions (4) and has the form

$$T_2 f = -\frac{\sqrt{\zeta^2 - 1}}{\pi} \iint_{|z| \leq 1} \left[ \frac{f(z)}{\sqrt{z^2 - 1}(z - \zeta)} - \frac{\overline{f(z)}}{\sqrt{1 - \bar{z}^2}(1 - z\zeta)} \right] dx dy.$$

Substituting  $\omega = T_2 f + \overline{\Phi(\zeta)}$  into equation (3), we obtain, for the determination of  $f(z)$ , the nonlinear singular integral equation

$$f - q_1 \frac{d}{d\zeta}(T_2 f) - q_2 \frac{d}{d\zeta}(\overline{T_2 f}) + \frac{d\overline{\Phi}}{d\zeta} = 0,$$

to which we apply the principle of contraction mappings in the spaces  $L_p$  ( $p = 2 + \varepsilon$ ,  $\varepsilon > 0$ ). A somewhat different path for proving the unique solvability of problem (3), (4) was proposed in the author's works (7, 8).

In particular, for  $k = 1$  we obtain unique solvability of the inverse mixed boundary-value problem for analytic functions both in the case of polygonality of the prescribed arcs  $L_i^1$  (which was proved in the author's papers (5, 6)), and

for curvilinear arcs, where the uniqueness theorem was not previously known. Especially simply, and for a broader class of plane filtration problems in an inhomogeneous medium, unique solvability is obtained in the case when the filtration coefficient  $\hat{k}$  does not depend explicitly on  $x, y$  and is a function only of the variables  $\varphi$  and  $\psi$ .

For example, when the compressibility of the medium is taken into account, the porosity  $m$  is assumed to be a function of the reduced pressure  $p = \gamma h = -\frac{\nu}{k_0}\varphi$ , where  $\nu = \rho g$  is the specific weight of the liquid ((<sup>2</sup>), p. 251). The filtration coefficient, as is known ((<sup>2</sup>), p. 16), is expressed by the formula  $\hat{k} = \frac{\nu}{\mu}C(m)$ , where  $C$  is the permeability of the soil and  $\mu$  is the viscosity of the liquid. Thus, for filtration in a compressible medium,  $k$  is a function only of  $\varphi$ . In some filtration problems in an inhomogeneous medium (for example, with weak inhomogeneity of the medium) the filtration coefficient can be represented in the form  $\hat{k} = \hat{k}_0 \frac{\psi - \psi_0}{\varphi - \varphi_0}$ , where  $\hat{k}_0, \varphi_0, \psi_0$  are certain parameters.

Thus, let  $k = k(\varphi, \psi)$ . Then the system of equations (2) is a Beltrami system of equations, any solution of which can be represented in the form  $x = x[\xi(\varphi, \psi), \eta(\varphi, \psi)]$ ,  $y = y[\xi(\varphi, \psi), \eta(\varphi, \psi)]$ , where  $\xi = \xi(\varphi, \psi)$ ,  $\eta = \eta(\varphi, \psi)$  is an arbitrary solution of this system effecting a homeomorphic mapping of the plane  $w = \varphi + i\psi$  onto the plane  $\zeta = \xi + i\eta$  (its determination leads to an integral equation solved by the method of successive approximations (<sup>9</sup>)), and  $x(\xi, \eta) + iy(\xi, \eta)$  is an analytic function of the variables  $\xi, \eta$ . Thus, in the plane of the variables  $\xi, \eta$  one obtains an inverse mixed boundary-value problem for an analytic function. Consequently, in this case the original problem of filtration in an inhomogeneous medium is reduced to an analogous problem in some homogeneous medium, whose unique solvability follows from the author's papers (<sup>5, 6</sup>).

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## REFERENCES

- <sup>1</sup> P. Ya. Polubarinova-Kochina, *Theory of the Motion of Ground Waters*, Moscow, 1952.
- <sup>2</sup> I. A. Charny, *Underground Hydrogasdynamics*, Moscow, 1963.
- <sup>3</sup> G. N. Polozhii, *Generalization of the Theory of Analytic Functions of a Complex Variable*, Kiev, 1965.
- <sup>4</sup> V. N. Monakhov, DAN, 156, No. 6 (1964).
- <sup>5</sup> V. N. Monakhov, in: *Collection: Functional Analysis and the Theory of Functions*, issue 1, Kazan, 1964.
- <sup>6</sup> V. N. Monakhov, DAN, 141, No. 4 (1961).

<sup>7</sup> V. N. Monakhov, *Proceedings of the Seminar on Inverse Boundary-Value Problems*, issue 2, Kazan, 1964.

<sup>8</sup> V. N. Monakhov, DAN, 164, No. 5 (1965).

<sup>9</sup> I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959.

*Note: Figure translations are in progress. See original paper for figures.*

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