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Abstract

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PROPAGATION OF UNSTEADY WAVES IN A ROTATING CHANNEL OF CONSTANT DEPTH

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A problem of the dynamical theory of tides is considered concerning unsteady waves caused in a rotating channel by certain initial disturbances; among the disturbing causes there may be underwater shocks leading to the occurrence of tsunami-type waves. The aim is to investigate the character of wave formations of this kind, due to the rectilinear walls of the channel, under hydrodynamic assumptions that simplify the problem.

If the system of equations describing the propagation of long waves on the surface of a rotating liquid layer of constant depth ([1]) is subjected to a Laplace transformation, then the system of transformed Euler equations and the continuity equation are written in the form

$$\nu U - 2\omega V + g\,\partial Z/\partial x = f_1(x,y), \qquad 2\omega U + \nu V + g\,\partial Z/\partial y = f_2(x,y),$$

$$\nu Z + h(\partial U/\partial x + \partial V/\partial y) = f(x,y), \tag{1}$$

where on the right-hand sides of each of equations (1) there is a function of the initial data of the problem.

The desired functions $U(x,y;\nu)$, $V(x,y;\nu)$, and $Z(x,y;\nu)$ of the intermediate problem must satisfy, in the unbounded strip $-\infty \le x \le \infty$, $0 \le y \le l$, the impermeability condition on the boundaries of the strip

$$V(x,y;\nu) = 0 \quad \text{for } y = 0,\ y = l, \tag{2}$$

and the condition of regularity of the functions U and Z as one moves away from the region of the initial disturbance and from the axis of rotation,

$$|U(x,y;\nu)| \to 0, \qquad |Z(x,y;\nu)| \to 0 \quad \text{as } |x| \to \infty. \tag{3}$$

System (1) permits the elimination from it of the functions U, V, and makes it possible to reduce the investigation to a boundary-value problem for an inhomogeneous Helmholtz equation with complex wave number σ/c

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} - \frac{\sigma^2}{c^2} Z = -\frac{\sigma^2}{\nu c^2} f(x,y) + \frac{1}{g}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) + \frac{2\omega}{g\nu}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \tag{4}$$

with inhomogeneous Poincaré boundary conditions

$$\nu\frac{\partial Z}{\partial y} - 2\omega\frac{\partial Z}{\partial x} + \frac{2\omega}{g} f_1(x,y) - \frac{\nu}{g} f_2(x,y) = 0 \quad \text{for } y = 0,\ y = l, \tag{5}$$

which is accompanied by an additional requirement of the type of a radiation condition

$$\left|\nu\frac{\partial Z}{\partial x} + 2\omega\frac{\partial Z}{\partial y} - \frac{\nu}{g} f_1(x,y) - \frac{2\omega}{g} f_2(x,y)\right| \to 0 \quad \text{as } |x| \to \infty. \tag{6}$$

The last condition (6) is a consequence of the regularity conditions (3) for the function U and imposes certain restrictions on the behavior of the solution at an infinitely remote point of the strip; the boundary conditions (5) are obtained from the impermeability conditions (2). As for the parameter $\sigma = (\nu^2 + 4\omega^2)^{1/2}$, it is determined through that branch of the root which corresponds to a positive real part in accordance with the requirements of the Laplace transform $\operatorname{Re}\nu > 0$.

By successively applying the Fourier transform to equation (4) and to the boundary conditions (5), the problem of finding the function $Z(x,y;\nu)$ is reduc—

reduces to finding its Fourier transform $S(y,k,\nu)$ from the ordinary differential equation

$$\frac{d^2 S}{dy^2} - \chi^2 S = -\frac{\sigma^2}{\nu c^2} F(y,k) - \frac{ik}{g} F_1(y,k) - \frac{2i\omega k}{g\nu} F_2(y,k) + \frac{1}{g}\frac{dF_2}{dy} - \frac{2\omega}{g\nu}\frac{dF_1}{dy} \tag{7}$$

with inhomogeneous boundary conditions of mixed type

$$\nu\frac{dS}{dy} + 2i\omega k S + \frac{2\omega}{g} F_1(y,k) - \frac{\nu}{g} F_2(y,k) = 0 \quad \text{for } y = 0,\ y = l; \tag{8}$$

here $F(y,k)$, $F_1(y,k)$, $F_2(y,k)$ are functions obtained by the Fourier transform of the right-hand sides in (1), $\chi^2 = k^2 + \sigma^2/c^2$.

The solution for $Z(x, y; \nu)$ is obtained with the aid of the integral theorem of Fourier inversion. Assuming the absolute integrability (in the Cauchy sense) of the functions $f(x, y)$, $f_1(x, y)$, $f_2(x, y)$ recovered in the process of the integral Fourier transforms and reversing the order of integration, one can obtain the function Z by the methods of Cauchy' s residue theory in such a way that the radiation condition (6) is satisfied; the final form of the expression for Z is established by contour integration of expansions into principal parts of the integrand dependences, which are single-valued (and regular in the corresponding half-planes $\operatorname{Im} k \gtreqless 0$) functions of the complex variable k. A rigorous justification of the nonformal constructions of equation (7), boundary conditions (8), and the method of obtaining the function $Z(x, y; \nu)$ can be traced in full if, for this purpose, one invokes the theory of the generalized Fourier integral (2).

Under assumptions concerning the proper smoothness of the functions of the initial state of the fluid, quantities of the form

$$\alpha_n(x) = \frac{1}{l}\int_0^l f(x, y)\cos\frac{\pi n y}{l}\,dy, \qquad \beta_n(x) = \frac{1}{l}\int_0^l f(x, y)\sin\frac{\pi n y}{l}\,dy \tag{9}$$

are taken as the amplitude coefficients of the individual waves under reflection of the initial disturbance from the rectilinear walls of the channel. With the aid of formulas (9), Z is expressed as follows:

$$\begin{aligned} Z(x, y; \nu) = {} & \frac{\omega}{c^2 \operatorname{sh} 2\omega l/c}\int_{-\infty}^{\infty}\int_0^l f(\xi, \eta)\ \operatorname{ch}\frac{2\omega}{c}(l - y - \eta)\exp\left[-\frac{\nu}{c}|x - \xi|\right]d\xi\, d\eta \\ & + \frac{\nu}{c^2}\sum_{n=1}^{\infty}\frac{1}{k_n^2}\int_{-\infty}^{\infty}P_n(x, \xi; \nu)\left[\frac{\pi^2 n^2}{l^2}\alpha_n(\xi)\cos\frac{\pi n y}{l} + \frac{4\omega^2}{c^2}\beta_n(\xi)\sin\frac{\pi n y}{l}\right]d\xi \\ & + \frac{2\omega}{c^2}\sum_{n=1}^{\infty}\frac{1}{k_n^2}\int_{-\infty}^{\infty}Q_n(x, \xi; \nu)\left[\alpha_n(\xi)\sin\frac{\pi n y}{l} + \beta_n(\xi)\cos\frac{\pi n y}{l}\right]d\xi \\ & + \frac{4\omega^2}{\nu c^2}\sum_{n=1}^{\infty}\sin\frac{\pi n y}{l}\int_{-\infty}^{\infty}P_n(x, \xi; \nu)\beta_n(\xi)\,d\xi, \qquad c^2 = gh, \end{aligned} \tag{10}$$

where here

$$P_n(x, \xi; \nu) = (\nu^2/c^2 + k_n^2)^{-1/2}\exp\left[-|x - \xi|(\nu^2/c^2 + k_n^2)^{1/2}\right],$$

$$Q_n(x, \xi; \nu) = \left\{\exp\left[-|x - \xi|\nu/c\right] - \exp\left[-|x - \xi|(\nu^2/c^2 + k_n^2)^{1/2}\right]\right\}\operatorname{sgn}(x - \xi)$$

are parametric functions of wave formation along the channel ($\operatorname{sgn} x$ denotes the Kronecker symbol). To satisfy conditions (3) under complex irrationality, that single-valued branch is selected which meets the requirements for σ.

Formula (10) determines the function Z from system (1) for the case when the initial elevation of the fluid surface in the channel is not accompanied by initial velocities; the remaining terms in (10), which contained

Both $f_1(x, y)$ and $f_2(x, y)$ can be expressed in an analogous way through the parametric functions $P_n(x, \xi; \nu)$ and $Q_n(x, \xi; \nu)$. The functions U and V are found directly by differentiating Z in view of equations (1).

The final form of the wave-elevation quantity is obtained from formula (10) by means of Mellin' s integral

$$\zeta(x, y, t) = \frac{\omega}{c} \frac{e^{-2\omega y/c}}{\operatorname{sh} 2\omega l/c} \int_0^l f(x - ct, \eta) e^{2\omega(l-\eta)/c} \, d\eta +$$

$$+ \frac{\omega}{c} \frac{e^{2\omega y/c}}{\operatorname{sh} 2\omega l/c} \int_0^l f(x + ct, \eta) e^{-2\omega(l-\eta)/c} \, d\eta + \sum_{n=1}^{\infty} \frac{1}{k_n^2} \left[\left(\frac{\pi^2 n^2}{l^2} \frac{\partial A_n}{\partial t} + 2\omega \frac{\pi n}{l} \frac{\partial B_n}{\partial x} \right) \right.$$

$$\left. \times \cos \frac{\pi n y}{l} + \left(2\omega \frac{\pi n}{l} \frac{\partial A_n}{\partial x} + \frac{4\omega^2}{c^2} \frac{\partial B_n}{\partial t} \right) \sin \frac{\pi n y}{l} \right] +$$

$$+ 4\omega^2 \sum_{n=1}^{\infty} \sin \frac{\pi n y}{l} \int_0^t B_n(x, \tau) \, d\tau, \qquad k_n^2 = \frac{4\omega^2}{c^2} + \frac{\pi^2 n^2}{l^2}. \tag{11}$$

The amplitude functions

$$A_n(x, t) = \frac{1}{c} \int_{x-ct}^{x+ct} J_0\left(k_n \sqrt{c^2 t^2 - |x - \xi|^2}\right) a_n(\xi) \, d\xi,$$

$$B_n(x, t) = \frac{1}{c} \int_{x-ct}^{x+ct} J_0\left(k_n \sqrt{c^2 t^2 - |x - \xi|^2}\right) \beta_n(\xi) \, d\xi, \tag{12}$$

represented in a source-like manner through the coefficients (9) and the Bessel function with the kernel characteristic of Riemann' s method, establish the formation of waves propagating in the channel. It presents no difficulty to determine the complete set of formulas of the type (11), (12), taking into account the combined action of the initial velocities $f_1(x, y)$, $f_2(x, y)$.

It follows from the expression for the wave elevation (11) that the unsteady waves due to the initial disturbance consist of two progressive Kelvin waves propagating in opposite directions along the channel; L. N. Sretenskii points out in his investigations ([3]) the possibility of the existence of Kelvin waves in a generalized form. The propagation of the progressive waves is accompanied by an infinite series of Poincaré waves, as indicated by the first infinite sum in (11); the wave formation that has developed in the channel is completed by accumulating sinusoidal Proudman waves, expressed in (11) by an integral.

If one sets $\omega = 0$, then the investigation, becoming considerably simpler, describes the propagation of a cylindrical sound wave arising inside an infinite waveguide from an instantaneous initial signal.

Using Schläfli' s trigonometric expansion from the theory of Bessel functions, applied to the amplitude function $A_n(x, t)$ in (12), the right-hand side of expression (11) can be reduced to the known formula for plane sound waves of the dynamical theory of sound,

$$\frac{1}{\pi c}\frac{\partial}{\partial t}\iint \frac{f(\xi, \eta)\, d\xi\, d\eta}{\sqrt{c^2 t^2 - (x-\xi)^2 - (y-\eta)^2}},$$

where the integration is extended over a segment of a circle of radius ct, and the signal function is extended evenly across the boundary $y = 0$.

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Note: Figure translations are in progress. See original paper for figures.