The application of Galerkin' s method to the solution of a mixed problem for a quasilinear hyperbolic equation

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Abstract

The Galerkin method is used to solve the mixed problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + f(t, x, u, u_t, u_x), \\ &u(0, x) = \varphi_0(x), \quad u_t(0, x) = \varphi_1(x), \\ &u(t, 0) = u(t, \pi) = 0, \end{aligned} \tag{1}$$

where $p(x)$ is a bounded, measurable function on $[0, \pi]$ such that $p(x) > 0$. The functions

$$\psi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx \quad (k = 1, 2, \dots)$$

are taken as the basis system. The existence and uniqueness of the generalized solution to problem (1), as well as the existence of an almost everywhere solution and a classical solution to this problem, are proven.

Bibliography: 9 items.

Full Text

Preamble

This work, following the developments in [7], considers the mixed problem for a nonlinear partial differential equation in the domain $Q = \{0 < t < T, 0 < x < \pi\}$. We seek a function $u(t, x)$ satisfying the equation:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(t, x, u, u_t, u_x) \quad (1)$$

subject to the initial conditions:

$$u(0,x) = \phi_0(x), \quad u_t(0,x) = \phi_1(x) \quad (2)$$

and the boundary conditions:

$$u(t,0) = u(t,\pi) = 0 \quad (3)$$

where $\rho(x) \geq \rho_0 > 0$ for $x \in [0,\pi]$. We assume that the eigenfunctions $\psi_k(x)$ of the corresponding linear operator satisfy the normalization condition $\int_0^\pi \rho(x)\psi_k(x)\psi_i(x)dx = \delta_{ki}$.

1. Construction of the Approximate Solution

We seek an approximate solution $u_n(t,x)$ using the Galerkin method in the form:

$$u_n(t,x) = \sum_{k=1}^{n} u_k^{(n)}(t)\psi_k(x) \quad (4)$$

Substituting this expression into equation (1) and projecting onto the basis $\{\psi_k(x)\}$, we obtain a system of ordinary differential equations for the coefficients $u_k^{(n)}(t)$:

$$\frac{d^2u_k^{(n)}}{dt^2} + \lambda_k u_k^{(n)} = \int_0^\pi f(t,x,u_n,\dots)\psi_k(x)dx \quad (5)$$

with initial conditions:

$$u_k^{(n)}(0) = \int_0^\pi \rho(x)\phi_0(x)\psi_k(x)dx, \quad \dot{u}_k^{(n)}(0) = \int_0^\pi \rho(x)\phi_1(x)\psi_k(x)dx \quad (6)$$

This system can be written in vector form as:

$$\ddot{u}^{(n)}(t) + A_n u^{(n)}(t) = f^{(n)}(t, u^{(n)}(t),\dots) \quad (7)$$

$$u^{(n)}(0) = \phi_0^{(n)}, \quad \dot{u}^{(n)}(0) = \phi_1^{(n)} \quad (8)$$

where A_n is a diagonal matrix of eigenvalues λ_k. The solution to this system on the interval $[0,T]$ can be represented by the integral equation:

$$u^{(n)}(t) = \cos(\sqrt{A_n}t)\phi_0^{(n)} + (\sqrt{A_n})^{-1}\sin(\sqrt{A_n}t)\phi_1^{(n)} + \int_0^t (\sqrt{A_n})^{-1}\sin(\sqrt{A_n}(t-\tau))f^{(n)}(\tau, u^{(n)}(\tau),\dots)d\tau \quad (9)$$

2. Convergence and Stability

To establish the existence of a solution on $[0,T]$, we impose growth conditions on the nonlinear term f. Specifically, we assume:

$$|f(t,x,u,u_t,u_x)| \leq a_1(t,x)|u| + a_2(t)|u_t| + a_3(t)|u_x| + a_4(t,x) \quad (10)$$

Using Gronwall-type inequalities and energy estimates, we demonstrate that the sequence of approximate solutions $u_n(t, x)$ is bounded in the appropriate Sobolev spaces. For the norm $\|u^{(n)}(t)\|^2 + \|\dot{u}^{(n)}(t)\|^2$, we derive the following estimate:

$$\|u^{(n)}(t)\|^2+\|\dot{u}^{(n)}(t)\|^2 \leq C \left(\|\phi_0^{(n)}\|^2 + \|\phi_1^{(n)}\|^2 + \int_0^t \beta_n(\tau) d\tau \right) \exp \left(\int_0^t \alpha_n(\tau) d\tau \right) \quad (11)$$

where $\alpha_n(t)$ and $\beta_n(t)$ are determined by the coefficients in (10).

Furthermore, we assume a Lipschitz condition for the uniqueness of the solution:

$$|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq \sum_{i=1}^{3} \kappa_i(t, K)|u_i - v_i| \quad (12)$$

Under these conditions, the sequence $u_n(t, x)$ converges to a function $u(t, x)$ that serves as a generalized solution to the original problem (1)-(3).

3. Regularity of the Solution

If the initial data $\phi_0(x)$ and $\phi_1(x)$ possess higher regularity, specifically $\phi_0 \in W_2^{(2)}$ and $\phi_1 \in W_2^{(1)}$, and satisfy the compatibility conditions $\phi_0(0) = \phi_0(\pi) = 0$, then the solution $u(t, x)$ belongs to the space $W_2^{(2)}(Q)$. In this case, the second-order derivatives exist in the sense of L_2, and the equation (1) is satisfied almost everywhere.

The proof utilizes the properties of the operator $B_n = \sqrt{A_n}$ and the convergence of the Galerkin approximations in stronger norms. We show that:

$$\|B_n^2 u^{(n)}(t)\|^2 + \|B_n \dot{u}^{(n)}(t)\|^2 \leq C_T \quad (14)$$

which ensures that the limit function $u(t, x)$ has the required square-integrable second derivatives.

Conclusion

The results obtained demonstrate the global existence and uniqueness of the solution to the nonlinear mixed problem. The method of semi-discretization (Galerkin method) provides not only a theoretical existence proof but also a basis for numerical implementation. The stability of the solution with respect to initial data and the right-hand side f follows from the derived energy inequalities. These findings extend classical results for linear hyperbolic equations to a broad class of nonlinearities relevant in mathematical physics.

Note: Figure translations are in progress. See original paper for figures.