A formula for the expansion of an arbitrary matrix-function in terms of the solution of a spectral problem

Authors: N. A. Aliev, M. L. Rasulov

Date: 1967-01-01T00:00:00+00:00

Abstract

The paper considers a spectral problem for a system of first-order ordinary linear differential equations of the form

$$\frac{dY}{dx} = AY + f, \tag{1}$$

$$\alpha Y(a, \lambda) + \beta Y(b, \lambda) = 0, \tag{2}$$

where

$$A = A(x, \lambda) = \lambda A(x) + \sum_{\nu=0}^{N} \lambda^{-\nu} A^{(\nu)}(x), \tag{3}$$

A, f, $A(x)$, $A^{(\nu)}(x)$, α, β, Y are square matrices of order n; λ is a complex parameter. An expansion formula for an arbitrary function of a certain class is obtained in terms of the solution to the spectral problem (1), (2), for the case when the arguments of the roots of the characteristic equation depend on the independent variable $x \in [a, b]$. Bibliography: 5 items.

Full Text

Preamble

This work, following the foundational approaches established in [**?**, **?**], investigates the asymptotic behavior of solutions to differential systems. We consider the system of equations:

$$\frac{dY}{dx} = A(x, \lambda)Y + f(x)$$

defined on the interval $[a, b]$, subject to the boundary conditions:

$$\alpha Y(a, \lambda) + \beta Y(b, \lambda) = 0$$

where λ is a complex parameter such that $\lambda = \lambda_1 + i\lambda_2$.

We assume the following conditions hold: 1. The matrix $A(x,\lambda)$ can be expanded in terms of the parameter λ. 2. The eigenvalues $\theta_k(x)$ of the lead matrix $A(x)$ are distinct for all $x \in [a,b]$. Specifically, we assume that for $i \neq j$, the condition Re $\lambda\theta_i(x) \neq$ Re $\lambda\theta_j(x)$ is satisfied, except perhaps at a finite number of points. 3. The system is regular in the sense of Birkhoff [**?**].

The formal solution to the homogeneous system can be represented as:

$$Y(x,\lambda) = (g^{(0)}(x) + \lambda^{-1}g^{(1)}(x) + \lambda^{-2}E(x,\lambda))e^{\lambda\int Q(t)dt}$$

where $g^{(0)}(x)$ is the matrix of eigenvectors of $A(x)$, satisfying $A(x)g^{(0)}(x) = g^{(0)}(x)Q(x)$. The subsequent terms $g^{(v)}(x)$ are determined by the recurrence relations:

$$g^{(v)}(x)Q(x) = A(x)g^{(v)}(x) + A^{(1)}(x)g^{(v-1)}(x) - \frac{d}{dx}g^{(v-1)}(x)$$

The error term $E(x,\lambda)$ remains bounded for sufficiently large $|\lambda|$.

For $x \in [a,b]$, we order the eigenvalues such that:

$$\text{Re } \lambda\theta_1(x) < \text{Re } \lambda\theta_2(x) < \cdots < \text{Re } \lambda\theta_n(x)$$

The general solution to the non-homogeneous equation (1) can be expressed using the fundamental matrix $Y(x,\lambda)$ and the Green' s function $G(x,\xi,\lambda)$:

$$Y(x,\lambda) = Y(x,\lambda)C + \int_a^b G(x,\xi,\lambda)f(\xi)d\xi$$

where the specific form of the constant vector C is determined by the boundary conditions (2). The solution to the boundary value problem is then:

$$Y(x,\lambda) = \int_a^b G(x,\xi,\lambda)f(\xi)d\xi$$

where $G(x,\xi,\lambda) = Y(x,\lambda)[\alpha Y(a,\lambda) + \beta Y(b,\lambda)]^{-1}$ ... as detailed in [**?**, **?**].

Following the methods of Tamarkin [**?**], we define the auxiliary functions:

$$\psi_k(x) = \int_a^x \theta_k(t)dt, \quad \phi_k(x) = \int_x^b \theta_k(t)dt$$

Under the assumption that Re $\lambda\psi_k(x) \neq 0$ for $k = 1,\dots,n$, we analyze the asymptotic behavior of the integral of the solution over a large contour in the λ-plane. As $|\lambda| \to \infty$, the integral of the solution $Y(x,\lambda)$ over the boundary of a region S in the complex plane yields:

$$\lim_{\nu\to\infty} \frac{1}{2\pi i}\oint Y(x,\lambda)d\lambda = A^{-1}(x)f(x)$$

This result demonstrates the convergence of the expansion in eigenfunctions of the boundary value problem (1)-(2) to the function $A^{-1}(x)f(x)$, consistent with the results established in [**?**, **?**].

References

1. Birkhoff G. D., Trans. Am. Math. Soc., 9, 1908, 219-231.
2. Tamarkin J. D., General problems of the theory of ordinary linear differential equations, Petrograd, 1917.
3. Tamarkin J. D., Math. Zeit., 27, 1927.
4. N. A. Aliev, M. L. Rasulov, Doklady Akad. Nauk SSSR, 119, No. 3, 450-453, 1958.
5. M. L. Rasulov, The contour integral method, Moscow, Nauka, 1964.

Note: Figure translations are in progress. See original paper for figures.