A GENERALIZED ABEL EQUATION AND AN EQUATION WITH A CAUCHY KERNEL

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Abstract

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A GENERALIZED ABEL EQUATION AND AN EQUATION WITH A CAUCHY KERNEL

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In the present note we investigate the normal solvability of the generalized Abel integral equation

$$M\varphi \equiv u(x)\int_a^x \frac{\varphi(t)\,dt}{(x-t)^\mu} + v(x)\int_x^b \frac{\varphi(t)\,dt}{(t-x)^\mu} + T\varphi = F(x) \tag{1}$$

and of the equation adjoint to it

$$M^*\psi \equiv \int_a^x \frac{v(t)\psi(t)}{(x-t)^\mu}\,dt + \int_x^b \frac{u(t)\psi(t)}{(t-x)^\mu}\,dt + T^*\psi = F_1(x), \tag{2}$$

using the connection of these equations with an equation with a Cauchy kernel. For $T=0$, equation (1) was first solved by K. D. Sakaliuk (^{5}). The stringent restrictions imposed in (^{5}) on u, v, F were weakened by L. F. Wolfersdorf (^{6}). Some results for the complete equation (1) were obtained (under stringent assumptions concerning T, u, v, F) by F. V. Chumakov (^{7}). In what follows all functions are assumed to be real-valued.

1°. Let $0<\alpha<1$ and $a \le x \le b$. We introduce the notation:

$$I^\alpha_{ax}\varphi \equiv \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}}, \qquad I^\alpha_{xb}\varphi \equiv \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)\,dt}{(t-x)^{1-\alpha}}; \tag{3}$$

$$A_\alpha\varphi \equiv \int_a^b \frac{\varphi(t)\,dt}{|x-t|^{1-\alpha}}, \qquad B_\alpha\varphi \equiv \int_a^b \frac{\operatorname{sign}(x-t)}{|x-t|^{1-\alpha}}\varphi(t)\,dt; \tag{4}$$

$$S\varphi \equiv \frac{1}{\pi}\int_a^b \frac{\varphi(t)}{t-x}\,dt; \tag{5}$$

$$r_a\varphi \equiv (x-a)\varphi(x), \qquad r_b\varphi \equiv (b-x)\varphi(x), \qquad r\varphi \equiv (x-a)(b-x)\varphi(x).$$

Lemma 1. The operators (3)–(5) are connected by the identities

$$B_\alpha\varphi \equiv -\operatorname{tg}\left(\frac{\alpha\pi}{2}\right) A_\alpha\left(\frac{1}{r^{\alpha/2}} S r^{\alpha/2}\varphi\right) \equiv -\operatorname{tg}\left(\frac{\alpha\pi}{2}\right) r^{\alpha/2} S\left(\frac{1}{r^{\alpha/2}} A_\alpha\varphi\right); \tag{6}$$

$$A_\alpha\varphi \equiv \operatorname{ctg}\left(\frac{\alpha\pi}{2}\right) B_\alpha\left(\frac{1}{r^{(1+\alpha)/2}} S r^{(1+\alpha)/2}\varphi\right) \equiv \operatorname{ctg}\left(\frac{\alpha\pi}{2}\right) r^{(1+\alpha)/2} S\left(\frac{1}{r^{(1+\alpha)/2}} B_\alpha\varphi\right); \tag{7}$$

$$I_{xb}^{\alpha}\varphi \equiv \cos(\alpha\pi)\, I_{ax}^{\alpha}\varphi + \sin(\alpha\pi)\, I_{ax}^{\alpha}\frac{1}{r_a^{\alpha}} S r_a^{\alpha}\varphi; \tag{8}$$

$$I_{ax}^{\alpha}\varphi \equiv \cos(\alpha\pi)\, I_{xb}^{\alpha}\varphi - \sin(\alpha\pi)\, I_{xb}^{\alpha}\frac{1}{r_b^{\alpha}} S r_b^{\alpha}\varphi; \tag{9}$$

$$r_b^{\alpha} S \frac{1}{r_b^{\alpha}} I_a^{\alpha}\varphi \equiv I_{ax}^{\alpha}\frac{1}{r_a^{\alpha}} S r_a^{\alpha}\varphi. \tag{10}$$

$$r_a^{\alpha} S \frac{1}{r_a^{\alpha}} I_{tb}^{\alpha}\varphi \equiv I_{xb}^{\alpha}\frac{1}{r_b^{\alpha}} S r_b^{\alpha}\varphi. \tag{11}$$

The proof of identities (6)–(11) is based on the fact that, after the order of integration is interchanged in the repeated integrals, the inner integrals obtained in (6)–(11) are easily expressed in terms of elementary functions*. Identities (6) –(11) are valid** if $\varphi \in L_p(\rho)$, where $L_p(\rho)$ is the class of functions summable on $[a,b]$ to the power $p > 1$ with weight $\rho(t)$, used in the theory of equations with Cauchy kernel (^{3}).

2°. With the aid of the relations (6)–(11), equations (1), (2) are reduced to conjugate equations with Cauchy kernel. We indicate three methods of reduction.

A. Writing (1)–(2) in the form

$$uI_{ax}^{1-\mu}\varphi + vI_{xb}^{1-\mu}\varphi + T\varphi = \frac{1}{\Gamma(1-\mu)}F; \tag{1'}$$

$$I_{ax}^{1-\mu}(v\psi) + I_{xb}^{1-\mu}(u\psi) + T^{*}\psi = \frac{1}{\Gamma(1-\mu)}F_1 \tag{2'}$$

and applying identities (8), (10) to (1′) and (9) to (2′), we obtain

$$a_1\Phi + a_2 r_b^{1-\mu} S \frac{1}{r_b^{1-\mu}}\Phi + K\Phi = F; \tag{12}$$

$$a_1\psi - \frac{1}{r_b^{1-\mu}} S r_b^{1-\mu} a_2\psi + K^*\psi = f_1, \tag{13}$$

where $K = TI_{at}^{-(1-\mu)}$, $\Phi = I_{ax}^{1-\mu}\varphi$, $f_1 = I_{ax}^{-(1-\mu)}F_1$***,
$a_1(x) = \Gamma(1-\mu)[u(x) - v(x)\cos\mu\pi]$,
$a_2(x) = \Gamma(1-\mu)\sin(\mu\pi)v(x)$.

We shall require of the operator T that the condition

$$TI_{at}^{-(1-\mu)} = T_1, \tag{14}$$

be satisfied, where T_1 is a completely continuous operator in the function space Φ under consideration (introduced below).

B. If, however, (9), (11) are applied to (1′) and (8) to (2′), then we obtain

$$b_1\chi - b_2 r_a^{1-\mu} S \frac{1}{r_a^{1-\mu}}\chi + K_1\chi = F,$$

$$b_1\psi + \frac{1}{r_a^{1-\mu}} S r_a^{1-\mu} b_2\psi + K_1^*\psi = f_2,$$

where $K_1 = TI_{tb}^{-(1-\mu)}$, $\chi = I_{xb}^{1-\mu}\varphi$, $f_2 = I_{xb}^{-(1-\mu)}F_1$,
$b_1(x) = \Gamma(1-\mu)[v(x) - u(x)\cos\mu\pi]$, $b_2(x) = \Gamma(1-\mu)\sin(\mu\pi)u(x)$, and, analogously to (14), one should require that

$$TI_{tb}^{-(1-\mu)} = T_1. \tag{14'}$$

C. Now writing (1)–(2) in the form

$$\frac{u+v}{2}A_{1-\mu}\varphi + \frac{u-v}{2}B_{1-\mu}\varphi + T\varphi = F; \tag{1''}$$

$$A_{1-\mu}\left(\frac{u+v}{2}\psi\right) + B_{1-\mu}\left(\frac{u-v}{2}\psi\right) + T^*\psi = F_1 \tag{2''}$$

* We obtain the inner integrals by solving Abel equations.

* For (8)–(11) one may use formula 3.228 from ([8]). For (6)–(7)
$\operatorname{sign}(\tau - x)\operatorname{ctg}(\alpha\pi/2)\varphi(x) - S\varphi = 0$,
$\operatorname{tg}(\alpha\pi/2)\psi(x) - S(\operatorname{sign}(t-\tau)\psi(t)) = 0$.

** Almost everywhere, if $p \le 1/\alpha$, and for all $x \in (a, b)$, if $p > 1/\alpha$.

*** $I_{ax}^{-(1-\mu)}$ is the fractional differentiation operator inverse to $I_{ax}^{1-\mu}$.

and applying (6) to (1″), (2″), we have

$$d_1\Omega - d_2 r^{(1-\mu)/2} S \frac{1}{r^{(1-\mu)/2}}\Omega + K_2\Omega = F,$$

$$d_1\psi + \frac{1}{r^{(1-\mu)/2}} S r^{(1-\mu)/2}\psi + K_2^*\psi = f_3,$$

where $K_2 = TA_{1-\mu}^{-1}$, $\Omega = A_{1-\mu}\varphi$, $f_3 = A_{1-\mu}^{-1}F_1$, $d_1 = (u+v)/2$, $d_2 = \operatorname{ctg}(\mu\pi/2)(u-v)/2$. Here one must require that

$$TA_{1-\mu}^{-1} = T_1. \tag{14”}$$

Using the identities (6)–(11), one can show that the requirements (14), (14′), (14″) are equivalent. A simple sufficient condition for the fulfillability of (14), (14′), (14″) is given by the following

Lemma 2. Let

$$T = \int_a^b T(x,t)\varphi(t)\,dt,$$

$$T(x,t) = \begin{cases} c_1(x,t)(x-t)^{-\nu_1}, & t < x, \\ c_2(x,t)(t-x)^{-\nu_2}, & t > x, \end{cases}$$

where $0 \le \nu_i < \mu$, $|\partial c_i/\partial t| < \mathrm{const}/|x-t|$, $i = 1, 2$. Then the kernels of the operators (14), (14′), (14″) can be represented in the form of the sum of a degenerate kernel and a kernel with a weak singularity.

Let us note that methods A and B differ from one another inessentially. They are preferable to method B, since in order to implement them one has to solve an Abel equation (the classical one), whereas in the third method one has to solve the more complicated equation $A_{1-\mu}\varphi = \Omega$.

3°. The operator M is completely continuous and therefore has no bounded regularizer. Consequently ([4]), M is not a Noether operator in the usual sense. However, for equations (1)–(2) Noether theorems may hold in special spaces.

The conclusions below are made on the basis of reducing equations (1)–(2) to equations with the Cauchy kernel. Let X and Y be spaces of functions in which, for equations (12)–(13) with the Cauchy kernel, Noether theorems hold. (For example, X and Y are allied (^{1},^{2}) Hölder classes of functions in (a, b), or conjugate spaces $\mathcal{L}_p(\rho)$, $\mathcal{L}_q(\rho^{1-q})$ (^{3}).) Suppose at first that X is such that there exist $p > 1$ and a weight $\rho(t)$ (as in (3), p. 12), for which $\mathcal{L}_p(\rho) \supset I_{ax}^{-(1-\mu)}(X)$. Denote then $B_X = I_{ax}^{-(1-\mu)}(X)$. The following is valid*.

Theorem 1. Let $u(x)$, $v(x) \in H^\lambda$, $\lambda > 1-\mu$, and $F(x) \in X$, $f_1(x) \in Y$. The Noether theorems for equations (1)–(2) hold if the solutions of (1) are sought in the space B_X, and the solutions of (2) in Y. The index of equation (1) in this case is equal to the index of equation (12).

In the following Lemma 3 we indicate a sufficient criterion for

$$I_{ax}^{-(1-\mu)}(X) \subseteq \mathcal{L}_p(\rho).$$

Lemma 3. Let $u(x)$, $v(x)$, $F(x) \in H^\lambda$, $\lambda > 1-\mu$, and let

$$\gamma(x) = \frac{1}{2\pi i} \ln \frac{G(x-0)}{G(x+0)},$$

where

$$G(x) = \begin{cases} (u - e^{\mu\pi i} v)(u - e^{-\mu\pi i} v)^{-1}, & x \in [a, b], \\ 1, & x \notin [a, b]. \end{cases}$$

* Under the assumption that $u^2(x) + v^2(x) \neq 0$.

If $-\mu < \gamma(a) < 1$, $-1 < \gamma(b) < \mu$, then all Hölder solutions Φ in (a, b) of equation (12) are representable in the form $\Phi = I_{ax}^{1-\mu}\varphi$, $\varphi(x) = (x-a)^{v_a}(b-x)^{v_b}\varphi_0(x)$, $\varphi_0(x) \in H^{\lambda+\mu-1}$, $v_a = \min(\mu - 1,\ \mu - 1 + \gamma(a))$, $v_b = \min(0, \gamma(b))$.

One can dispense with the smoothness requirement on $u(x)$ and $v(x)$ if, for (1), solutions are allowed in the class of generalized functions. Let now $X = \mathcal{L}_p(\rho)$, $p > 1$, and $\rho(t) = (b-t)^{-p(1-\mu)}\rho_0(t)$, where the weight $\rho_0(t)$ is introduced according to the function $G(t)$, following (^{3}), p. 85, and let B be the space of generalized fractional derivatives of order $1-\mu$ of functions from $\mathcal{L}_p(\rho)$ (functionals $\varphi = \Phi^{(1-\mu)}$ on the class of basic functions $\Psi(x)$ of the form $\Psi(x) = I_{xb}^{1-\mu}\psi$, $\psi \in \mathcal{L}_q(\rho^{1-q})$):

$$(\varphi, \Psi) = (I_{ax}^{-1+\mu}\Phi, \Psi) = (\Phi, I_{xb}^{-1+\mu}\Psi) = (\Phi, \psi).$$

Defining in the proper way the operators (3)–(5) in the space B, it is not difficult to show that identities (6)–(11) are also valid for $\varphi \in B$, and, consequently, the reduction to equation (12) remains in force.

The principal result is given by the following.

Theorem 2. Let $u(x)$, $v(x)$ be continuous on $[a, b]$, $f_1 \in \mathcal{L}_q(\rho^{1-q})$, $F(x) \in \mathcal{L}_p(\rho)$, and let T be a completely continuous operator from B into $\mathcal{L}_p(\rho)$. Then, for equations (1)–(2), Noether' s theorems hold in the spaces B and $\mathcal{L}_q(\rho^{1-q})$, respectively.

Theorem 3. Let R_s be a regularizer of equation (12). Then the operator $R = I_{ax}^{-1+\mu} R_s$ is a regularizer (both left and right) of the operator M, and regularization on the left leads to an equation regular in B: $RM\varphi \equiv \varphi + T_B\varphi = RF$, while regularization on the right leads to an equation regular in $\mathcal{L}_p(\rho)$: $MR\Phi = \Phi + T_L\Phi = F$; T_B and T_L are completely continuous operators in B and in $\mathcal{L}_p(\rho)$, respectively.

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Note: Figure translations are in progress. See original paper for figures.