ON FUNDAMENTAL SOLUTIONS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract

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ON FUNDAMENTAL SOLUTIONS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Let D be a domain of $(n+1)$-dimensional Euclidean space adjoining the hyperplane $x_{n+1}=0$. In this domain consider a linear differential operator of order $2m$ of B-elliptic type $(^1)$

$$\mathcal{L}(D_x, B_{x_{n+1}}) = \sum_{\nu=0}^{m} \sum_{i=0}^{2m-2\nu} \sum_{j_1,\dots,j_i=1}^{n} A^{\nu}_{j_1,\dots,j_i}(x) \frac{\partial^i}{\partial x_{j_1},\dots,\partial x_{j_i}} B^{\nu}_{x_{n+1}}. \qquad (1)$$

Here, as usual, $B_{x_{n+1}}$ denotes the Bessel differential operator

$$B_{x_{n+1}} = \frac{\partial^2}{\partial x^2_{n+1}} + \frac{k}{x_{n+1}} \frac{\partial}{\partial x_{n+1}},$$

where $k>0,\ x_{n+1} \geqslant 0$.

In paper $(^2)$ a fundamental solution of the homogeneous operator (1) was constructed in the case of constant coefficients. In the present note we construct a fundamental solution for the nonhomogeneous operator (1) with constant coefficients. In addition, we prove the existence in the small of a fundamental solution of this operator with variable coefficients. The case $m=1$ was studied earlier by A. Weinstein $(^6)$ and others.

1. Let $f(x) = f(x_1,\dots,x_n,x_{n+1})$ belong to the class C_2 and be finite in the half-space R^*_{n+1} $(x_{n+1} \geqslant 0)$. Then this function admits the following expansion in weighted plane waves:

$$f(x) = c_1(u,k,\nu)\Delta_B^{(\nu+\gamma)/2} \int_{R^*_{n+1}} f(y) \left(\int_{\Omega_+} T^y_x |x\cdot\omega|^{\nu}_B \omega^k_{n+1}\, d\omega \right) y^k_{n+1}\, dy, \qquad (2)$$

where Ω_+ is the hemisphere $\sum_1^{n+1}\omega_i^2 = 1$, $\omega_{n+1} \geqslant 0$; T_x^y is the generalized translation operator (^{3})

$$T_x^y f(x) = c(k)\int_0^\pi f\Big(y_1 - x_1, \dots, y_n - x_n, \sqrt{x_{n+1}^2 + y_{n+1}^2 - 2x_{n+1}y_{n+1}\cos\alpha}\Big)\sin^{k-1}\alpha\, d\alpha;$$

Δ_B is the Beltrami operator

$$\Delta_B = \sum_1^n \frac{\partial^2}{\partial x_i^2} + B_{x_{n+1}}.$$

An essential role in formula (2) is played by the integral of the form

$$|x\cdot\omega|_B^\nu = \int_0^\pi \left|\sum_1^n x_i\omega_i + x_{n+1}\omega_{n+1}\cos\alpha\right|^\nu \sin^{k-1}\alpha\, d\alpha,$$

where $\gamma = n+1+k$ and ν is the addition to the number γ up to an even number.

In the case of integer k, expansion (2) has the form

$$f(x) = c_2(n,k,\nu)\Delta_B^{(\nu+\gamma)/2}\int_{R_{n-1}^+} f(y)\left(\int_{\Omega_+} T_x^y|x\cdot\omega|_B^\nu \ln|x\cdot\omega|_B\omega_{n+1}^k\, d\omega\right) y_{n+1}^k\, dy. \tag{3}$$

To obtain formulas (2) and (3), the method of F. John is used [4].

2. Let $\mathcal{L}(D_x, B)$ be a B-elliptic operator with constant coefficients. The problem of finding the fundamental solution of the operator $\mathcal{L}(D_x, B_{x_{n+1}})$ is equivalent to solving the inhomogeneous equation
$$\mathcal{L}(D_x, B_{x_{n+1}})u = f(x) \tag{4}$$
for an arbitrary right-hand side $f(x)$. In view of the validity of expansions (2) and (3), it is sufficient to solve the equation
$$\mathcal{L}(D_x, B_{x_{n+1}})u = |x\cdot\omega|_B^\nu.$$

We seek the solution of this equation in the form

$$u = \int_0^\pi v\left(\sum_1^n x_i\omega_i + x_{n+1}\omega_{n+1}\cos\alpha\right)\sin^{k-1}\alpha\, d\alpha.$$

As a result, for the function $v(\xi)$ we obtain an ordinary differential equation, whose solution is written with the aid of the Duhamel integral. Then the fundamental solution of equation (4) will have the form

$$K(x) = c\Delta_B^{(\nu+\gamma)/2}\int_{\Omega_+}\int_0^\pi\int_0^\xi\int_C \frac{e^{(\xi-\tau)\lambda}}{\mathcal{L}(\omega\lambda)}\, d\lambda\, g(\tau)\, d\tau\, \sin^{k-1}\alpha\, d\alpha\, \omega_{n+1}^k\, d\omega, \tag{5}$$

where

$$\xi = \sum_1^n x_i\omega_i + x_{n+1}\omega_{n+1}\cos\alpha,$$

$$g(\tau) = \begin{cases} |\tau|^\nu \ln|\tau|, & \text{if } \gamma \text{ is even,} \\ |\tau|^\nu, & \text{in all other cases.} \end{cases}$$

Here C is a contour in the complex λ-plane enclosing all roots of the equation $\mathcal{L}(\omega\lambda) = 0$. The existence of such a standard contour for all $\omega = (\omega_1, \dots, \omega_{n+1})$ is ensured by the B-ellipticity of the operator $\mathcal{L}(D_x, B_{x_{n+1}})$. To obtain a fundamental solution with a singularity at an arbitrary point z, one must apply the shift operator T_x^z to the function $K(x)$. In the case of the homogeneous operator $\mathcal{L}(D_x, B_{x_{n+1}})$, the contour integral in formula (5) is readily computed. As a result, the fundamental solution has the form

$$K(x) = c\Delta_B^{(\nu+\gamma)/2} \int_{\Omega_+}\int_0^\pi |\xi|^{2m+\nu} \sin^{k-1}\alpha \, d\alpha \, \frac{\omega_{n+1}^k}{\mathcal{L}(\omega)} \, d\omega,$$

if γ is not an even number. If, however, γ is an even number, then

$$K(x) = c\Delta_B^{\gamma/2} \int_{\Omega_+}\int_0^\pi \xi^{2m} \ln|\xi| \sin^{k-1}\alpha \, d\alpha \, \frac{\omega_{n+1}^k}{\mathcal{L}(\omega)} \, d\omega.$$

3. To describe the structure of the fundamental solution (5), we expand $e^{(\xi-\tau)\lambda}$ in a series and successively bring, in each term, the operator Δ_B under the integral sign and then integrate by parts. Then, for the case when γ is not even, we obtain

$$K(x) = r^{2m-\gamma} \sum_{i=0}^\infty r^i \Omega_i\left(\frac{x}{r}\right),$$

where $\Omega_i(x/r)$ are infinitely differentiable functions, even with respect to the variable x_{n+1}. In the case of even γ we have

$$K(x) = r^{2m-\gamma} \sum_{i=0}^\infty r^i \Omega_i\left(\frac{x}{r}\right) + W(x)\ln r,$$

where $W(x)$ is a regular solution of the equation $\mathcal{L}(D_x, B_{x_{n+1}})W = 0$. In the case of a homogeneous operator we have

$$K(x) = r^{2m-\gamma}\Omega\left(\frac{x}{r}\right), \tag{6}$$

$$K(x) = r^{2m-\gamma}\Omega\left(\frac{x}{r}\right) + q_{2m-\gamma}(x)\ln r, \tag{7}$$

where $q_{2m-\gamma}(x)$ is a homogeneous polynomial of degree $2m-\gamma$, even in x_{n+1}. From formulas (6) and (7) we obtain the estimates

$$\left|D_x^i B_{x_{n+1}}^j K(x)\right| \le \text{const} \cdot r^{2m-\gamma-i-2j}.$$

If γ is even and $i+2j \le 2m-\gamma$, then

$$\left|D_x^i B_{x_{n+1}}^j K(x)\right| \le \text{const} \cdot r^{2m-\gamma-i-2j}(1+|\ln r|).$$

Inside the domain D ($x_{n+1}>0$), the fundamental solution and its derivatives have the same singularity as the fundamental solutions of ordinary elliptic equations.

4. We now consider the operator $\mathcal{L}(D_x, B_{x_{n+1}})$, whose coefficients $A^{\nu}_{j_1,\dots,j_i}(x)$ are i times continuously differentiable in the domain D with respect to the arguments $x_1,\dots,x_n$ and withstand ν applications of the operator $B_{x_{n+1}}$. Let $u(x)$ and $v(x)$ be two functions, $2m$ times continuously differentiable in the domain D with respect to the arguments $x_1,\dots,x_n$ and withstanding m applications of the operator $B_{x_{n+1}}$. We denote the class of such functions by C_B^{2m}. Then the following Green formula is valid:

$$\int_D (v\mathcal{L}u - u\overline{\mathcal{L}}v)\, x_{n+1}^k\, dx = \int_S R[u,v]\, x_{n+1}^k\, dS, \tag{8}$$

where

$$\overline{\mathcal{L}}v = \sum_{\nu=0}^{m} \sum_{i=0}^{2m-2\nu} (-1)^i \sum_{j_1,\dots,j_i=1}^{n} \frac{\partial^j}{\partial x_{j_1}\cdots\partial x_{j_i}} B^{\nu}\left(A^{\nu}_{j_1,\dots,j_i}(x)v\right),$$

and $R[u,v]$ is a bilinear differential operator.

For any point $z\in D$ we define the fundamental solution $K(x,z)$ with pole at the point z as such a function of x that the identity

$$v(z) = \int_D \overline{\mathcal{L}}(v)K(x,z)x_{n+1}^k\, dx + \int_S R[K(x,z),v(x)]x_{n+1}^k\, dS \tag{9}$$

holds, where $v(x)\in C_B^{2m}$. Following the Levi method (^{5}), we introduce into consideration the operator $\mathcal{L}^z$, a homogeneous operator with constant coefficients, which is obtained from the operator $\mathcal{L}(D_x, B_{x_{n+1}})$ by retaining the highest-order terms and freezing the coefficients at the point z. Let $K^+(x,z)$ be the fundamental solution of the operator $\mathcal{L}^z$ with pole at the point z. We shall seek the function $K(x,z)$ in the form

$$K(x,z) = K^{+}(x,z) + \int_{D} K^{+}(x,y)u(y,z)y_{n+1}^{k}\,dy. \tag{10}$$

Using Green' s formula (8) and identity (9), with respect to the function $u(y,z)$, we obtain an integral equation with a kernel having a weak singularity, whence it follows that, for a sufficiently small domain D, there exists a unique solution of this equation. Then formula (10) gives the desired fundamental solution $K(x,z)$ of the operator $\mathcal{L}(D_x, B)$.

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Note: Figure translations are in progress. See original paper for figures.

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