A NOTE ON MASS FORMULAS

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Abstract

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A NOTE ON MASS FORMULAS

IN THE THEORY OF UNITARY SYMMETRY

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It is known that, in order to obtain mass formulas in the $SU(3)$ scheme, it is sufficient to add to the invariant mass operator a perturbation with suitable tensor properties. In particular, the Gell-Mann–Okubo formula ([1])*

$$(\Xi + N)/2 = (3\Lambda + \Sigma)/4 \tag{1}$$

arises in the case in which the indicated perturbation transforms as the (33)-component of an octet.

The analogue of (1) for the masses of the quarks p, n, λ is the relation

$$p = n \neq \lambda. \tag{2}$$

The aim of the present note is to give a derivation of formulas of type (1) without relying on concrete expressions for the mass operator connected with perturbation theory. For us it will be essential only that this operator is diagonal in the isotopic spin I and hypercharge Y.

Let us introduce for consideration a "rotating" triplet of quarks

$$q' = \begin{pmatrix} p' \\ n' \\ \lambda' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p \\ n \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ n\cos\theta + \lambda\sin\theta \\ -n\sin\theta + \lambda\cos\theta \end{pmatrix}. \tag{3}$$

The transformation (3) belongs to the group $SU(3)$, and therefore in the primed coordinate system the components of the multiplets are characterized by the same set of quantum numbers as in the original system. At the same time, obviously,

$$Q' = Q, \quad U'(U' + 1) = U(U + 1), \tag{4}$$

where Q is the electric charge; $U(U+1)$ is the invariant of the U-spin group acting in the subspace (n, λ). It is clear that no definite mass can be assigned to the quarks n' and λ'. However, one may consider the mean values of the mass in these states, putting**

$$\overline{m}(n') = n\cos^2\theta + \lambda\sin^2\theta; \qquad \overline{m}(\lambda') = n\sin^2\theta + \lambda\cos^2\theta. \tag{5}$$

In the coordinate system corresponding to $\theta = \pi/4$, we shall have

$$\overline{m}(n') = \overline{m}(\lambda') = (n+\lambda)/2. \tag{6}$$

Since the n'- and λ'-quarks form a doublet with respect to U'-spin, the equality (5) means that the mean mass and U'-spin commute with each other. It is natural to suppose that, also in higher $SU(3)$ multiplets constructed from q'-quarks, the mean masses of states belonging to one

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* Below, the masses of particles are denoted by the same symbols as the particles themselves.

** For the equalities (5) and similar relations in other multiplets to be fulfilled, the assumption of diagonality of the mass operator in I and Y is essential. Indeed,

$$\overline{m}(n') = \langle n\cos\theta + \lambda\sin\theta \mid m \mid n\cos\theta + \lambda\sin\theta\rangle = n\cos^2\theta + \lambda\sin^2\theta,$$

if $\langle\lambda|m|n\rangle = \langle n|m|\lambda\rangle = 0$.

U'-multiplet are the same. Then certain relations must exist between the masses of **real** particles belonging to the same U'-multiplets. Below, using the example of the baryon octet $1/2^+$ and the decuplet of baryon resonances $3/2^+$, we shall show that these relations are equivalent to the usual mass formulas.

It is not difficult to see that, after the transformation (3) with $\theta = \pi/4$, the baryon octet takes the form

$$\|B_b'^a\| = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{1}{\sqrt{6}}\Lambda & \frac{\Sigma^+ - p}{\sqrt{2}} & \frac{\Sigma^+ + p}{\sqrt{2}} \\ \frac{\Sigma^- - \Xi^-}{\sqrt{2}} & -\frac{1}{2}\left(\frac{\Sigma^0}{\sqrt{2}} + \frac{1}{\sqrt{6}}\Lambda\right) - \frac{1}{2}(n+\Xi^0) & \frac{1}{2}\sqrt{\frac{3}{2}}\Lambda - \frac{\Sigma^0}{2\sqrt{2}} + \frac{1}{2}(n-\Xi^0) \\ \frac{\Sigma^- + \Xi^-}{\sqrt{2}} & \frac{1}{2}\sqrt{\frac{3}{2}}\Lambda - \frac{\Sigma^0}{2\sqrt{2}} - \frac{1}{2}(n-\Xi^0) & -\frac{1}{2}\left(\frac{\Sigma^0}{\sqrt{2}} + \frac{1}{\sqrt{6}}\Lambda\right) + \frac{1}{2}(n+\Xi^0) \end{pmatrix}. \tag{7}$$

Now from (7) let us isolate the U'-multiplets:

$$\frac{1}{2}\left(\Lambda+\sqrt{3}\,\Sigma^0\right)\equiv S^0 \qquad (U'=0);$$

$$\left(\frac{\Sigma^+-p}{\sqrt{2}},\frac{\Sigma^++p}{\sqrt{2}}\right)\equiv D^+, \qquad \left(\frac{\Sigma^--\Xi^-}{\sqrt{2}},\frac{\Sigma^-+\Xi^-}{\sqrt{2}}\right)\equiv D^- \qquad \left(U'=\frac{1}{2}\right);$$

$$\left(\frac{1}{2}\sqrt{\frac{3}{2}}\Lambda-\frac{\Sigma^0}{2\sqrt{2}}-\frac{1}{2}(n-\Xi^0),\ -\frac{n+\Xi^0}{\sqrt{2}},\ \frac{1}{2}\sqrt{\frac{3}{2}}\Lambda-\frac{\Sigma^0}{2\sqrt{2}}+\frac{1}{2}(n-\Xi^0)\right)\equiv(T_1^0,T_0^0,T_{-1}^0)\equiv T^0 \qquad (U'=1). \tag{8}$$

It is clear that in the doublets D^+ and D^- the average mass in states with different values of U_3 is the same, and moreover

$$\bar{m}(D^+)=(\Sigma^++p)/2, \qquad \bar{m}(D^-)=(\Xi^-+\Sigma^-)/2. \tag{9}$$

In the triplet T^0, obviously, the average masses of the first and third components are equal:

$$\bar{m}(T_1^0)=\bar{m}(T_{-1}^0)=\frac{3}{8}\Lambda+\frac{\Sigma^0}{8}+\frac{(n+\Xi^0)}{4}. \tag{10}$$

From the condition

$$\bar{m}(T_1^0)=\bar{m}(T_{-1}^0)=\bar{m}(T_0^0) \tag{11}$$

we find

$$(n+\Xi^0)/2=(3\Lambda+\Sigma^0)/4, \tag{12}$$

which coincides with formula (1) for neutral baryons.*

Thus the Gell-Mann—Okubo formula for baryon masses has been obtained by us without using any approximate expression for the mass operator and, in particular, without the assumption of its octet nature. However, in deriving relation (12) we ignored the renormalization of the baryon wave functions, assuming that the coefficients with which these functions enter the octet B_b^a remain unchanged after the interaction is switched on. Evidently, such an approximation corresponds to first order of perturbation theory.

Comparing this scheme with the usual approach, we conclude that the average masses of the U'-multiplets into which the octet decomposes in the chosen coordinate system must also be connected by a formula of the type (1):

$$[\bar{m}(D^+) + \bar{m}(D^-)]/2 = [3\bar{m}(S^0) + \bar{m}(T^0)]/4, \tag{13}$$

whence, taking into account (8), (9), (10), (12), we find:

$$\frac{\Sigma^+ + p + \Xi^- + \Sigma^-}{4} = \frac{3}{4}\left(\frac{\Lambda + 3\Sigma^0}{4}\right) + \frac{1}{4}\left(\frac{n + \Xi^0}{2}\right). \tag{14}$$

* According to (2), the Gell-Mann–Okubo relation must be applied precisely in the form (12).

This relation* is in good agreement with experiment (the left-hand side is 1161.4 ± 0.1 MeV, the right-hand side 1161.5 ± 0.2 MeV).

Let us note that if, on the basis of (12), in formula (14) one makes the replacement
$(n + \Xi^0)/2 \to \frac{1}{3}[n + \Xi^0 + (3\Lambda + \Sigma^0)/4]$, then as a result one obtains the relation

$$\frac{1}{2}\left[\frac{\Sigma^+ + p}{2} + \frac{\Xi^- + \Sigma^-}{2}\right] = \frac{1}{4}\left[3\left(\frac{\Lambda + 3\Sigma^0}{4}\right) + \frac{1}{3}\left(n + \Xi^0 + \frac{3\Lambda + \Sigma^0}{4}\right)\right], \tag{15}$$

which may be interpreted as the Gell-Mann–Okubo formula for the centers of gravity of the ordinary U-multiplets (in this case the "masses" of the mixed states are determined according to the footnote to formulas (5)). It is not difficult to see that the centers of gravity of the multiplets corresponding to V-spin also obey a formula analogous to (15).

Now let us turn to the decuplet $3/2^+$. The transformation (3) with $\theta = \pi/4$ leads to the following U'-multiplets:

1) $U' = \frac{3}{2}$:

$$\Omega^{-1} = \frac{1}{2\sqrt{2}}\left(\Omega^- - \Delta^- - \sqrt{3}\,\Xi_\delta^- + \sqrt{3}\,\Sigma_\delta^-\right),$$

$$\Xi_\delta^{-1} = \frac{1}{2\sqrt{2}}\left(\sqrt{3}\,\Omega^- + \sqrt{3}\,\Delta^- - \Xi_\delta^- - \Sigma_\delta^-\right),$$

$$\Sigma_\delta^{-1} = \frac{1}{2\sqrt{2}}\left(\sqrt{3}\,\Omega^- - \sqrt{3}\,\Delta^- - \Xi_\delta^- + \Sigma_\delta^-\right),$$

$$\Delta^{-1} = \frac{1}{2\sqrt{2}}\left(\Omega^- + \Delta^- + \sqrt{3}\,\Xi_\delta^- + \sqrt{3}\,\Sigma_\delta^-\right);$$

2) $U' = 1$:

$$\Delta^{0'} = \frac{1}{2}\Delta^0 + \frac{1}{\sqrt{2}}\Sigma^0_\delta + \frac{1}{2}\Xi^{0'}_\delta, \qquad \Xi^{0'}_\delta = \frac{1}{2}\Delta^0 - \frac{1}{\sqrt{2}}\Sigma^0_\delta + \frac{1}{2}\Xi^0_\delta, \tag{16}$$

$$\Sigma^{0'}_\delta = -\frac{1}{2}\Delta^0 + \frac{1}{\sqrt{2}}\Xi^0_\delta;$$

3) $U' = \frac{1}{2}$:

$$\Delta^{+'} = \frac{1}{\sqrt{2}}\left(\Delta^+ + \Sigma^+_\delta\right), \qquad \Sigma^{+'}_\delta = \frac{1}{\sqrt{2}}\left(-\Delta^+ + \Sigma^+_\delta\right);$$

4) $U' = 0$:

$$\Delta^{++}.$$

From this, applying the already known procedure, we shall have

$$\Omega^- - \Xi^-_\delta = \Sigma^-_\delta - \Delta^-, \qquad \Xi^0_\delta - \Sigma^0_\delta = \Sigma^0_\delta - \Delta^0. \tag{17}$$

In addition (cf. (13)–(15)),

$$(\Omega^- + \Delta^- + 3\Xi^-_\delta + 3\Sigma^-_\delta)/8 - (\Delta^0 + \Xi^0_\delta)/2$$

$$= (\Delta^0 + \Xi^0_\delta - \Sigma^+_\delta - \Delta^+)/2 = (\Sigma^+_\delta + \Delta^+)/2 - \Delta^{++}. \tag{18}$$

It is easy to see that, when isotopic splitting is neglected, relations (17)–(18) are equivalent to the well-known interval rule

$$\Omega - \Xi_\delta = \Xi_\delta - \Sigma_\delta = \Sigma_\delta - \Delta. \tag{19}$$

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2. S. Okubo, Electromagnetic Mass Differences in the Unitary Symmetry Model, Preprint UR-875-26, Rochester, 1964.

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* In the approximation $p = n = N$, $\Xi^0 = \Xi^- = \Xi$, $\Sigma^0 = \Sigma^- = \Sigma^+ = \Sigma$, formula (14), obviously, goes over into (1). Strictly speaking, we have no right to take electromagnetic mass differences into account in formulas (12), (14), and similar formulas, by virtue of the assumption of isotopic invariance of the mass operator.

Note: Figure translations are in progress. See original paper for figures.