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Dissipativity and periodic solutions of second order nonsymmetric systems

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Abstract

Sufficient conditions are established for the dissipativity of systems

$$\dot{x} = y, \quad \dot{y} = -yf(x, y, t) - g(x) + e(t), \tag{1}$$

$$\dot{x} = y - f(x, y, t), \quad \dot{y} = -g(x) \tag{2}$$

in the case where the function f satisfies substantially different constraints for $x > x_0 > 0$ and for $x < -x_0$, such that the "total energy" $y^{2/2} + \int g(x)\,dx$ of the system may increase during motion in the region $x < -x_0$, but necessarily decreases during motion in the region $x > x_0$. Bibliography: 4 items.

Full Text

Preamble

This study investigates the stability and boundedness of solutions for second-order nonlinear differential equations. We consider the system defined by:

$$\dot{x} = y$$
$$\dot{y} = -yf(x, y) - g(x) + e(t)$$

where $f(x, y)$, $g(x)$, and $e(t)$ are continuous functions. Following the methodology established in [?], we assume that for $|x| > x_0 > 0$, the damping term satisfies $f(x, y) > \phi(x)$, where $m > 0$. Furthermore, we assume $g(x) > mx$ for $x > x_0$ and $g(x) < mx$ for $x < -x_0$.

Let $G(x) = \int_0^x g(z)dz$ and $\Phi(x) = \int_0^x \phi(z)dz$. We assume that as $|x| \to \infty$, $G(x) \to \infty$. The growth of the integrated damping function $\Phi(x)$ is constrained by the potential function $G(x)$ such that:

$$\Phi(x) < a\sqrt{G(x)} + c_1, \quad x < -x_0$$
$$\Phi(x) > b\sqrt{G(x)} + c_2, \quad x > x_0$$

where a and b are constants satisfying specific stability criteria.

1. Qualitative Analysis of the Phase Trajectories

To analyze the behavior of the system, we introduce the transformation $u = y + \Phi(x)$. The system equations can then be rewritten as:

$$\dot{x} = u - \Phi(x)$$
$$\dot{u} = (u - \Phi(x))[\phi(x) - f(x, u - \Phi(x))] + e(t) - g(x)$$

For $|x| > x_0$, we consider the simplified comparison system:

$$\dot{x} = u - \Phi(x)$$
$$\dot{u} = -g(x) \pm m$$

where m accounts for the bounded perturbation $e(t)$ and the residual damping terms. We define the auxiliary functions $v_{\pm}(x) = \sqrt{2(G(x) \mp mx + c_4)}$, which represent the energy levels of the system.

The phase plane is divided into several regions $D_1, \dots, D_8$ based on the relationship between u, $\Phi(x)$, and the curves $v_{\pm}(x)$. For instance, in region D_1 (where $x < -x_0$ and $u < \Phi(x)$), the trajectories are governed by the interaction between the potential energy $G(x)$ and the integrated damping $\Phi(x)$.

2. Construction of the Lyapunov Function

To prove the ultimate boundedness of the solutions, we construct a piecewise Lyapunov function $V(x, u)$. In the regions where $|x| > x_0$, we utilize the function:

$$U(v, u, c) = \ln(v^2 - cuv + u^2) - \gamma(c) \arctan\left(\frac{2u - cv}{v\gamma(c)}\right)$$

where $\gamma(c) = \sqrt{4 - c^2}$. This functional form is derived from the quadratic structure of the energy equations under the linear approximation of the damping constraints.

The total Lyapunov function $V(x, u)$ is defined across the regions D_i by shifting U by appropriate constants η to ensure continuity or controlled jumps across the boundaries. For example: - In D_2: $V(x, u) = U(v_-, u, -a)$ - In D_5: $V(x, u) = U(v_+, u, \beta) + \eta$

By calculating the derivative $\dot{V}$ along the trajectories of the system, we demonstrate that for sufficiently large values of $x^2 + u^2$, the condition $\dot{V} < -c < 0$ holds. This implies that all trajectories eventually enter and remain within a bounded region Q in the phase plane.

3. Global Stability and Boundedness Results

The analysis shows that if the damping $f(x, y)$ grows sufficiently fast relative to the restoring force $g(x)$, specifically satisfying the square-root growth conditions relative to $G(x)$, then the system is dissipative in the sense of Levinson.

Specifically, we have shown that: 1. All solutions $x(t), y(t)$ are defined for all $t > t_0$. 2. There exists a compact set K such that for any initial condition, the trajectory enters K after a finite time. 3. If $e(t)$ is periodic with period T, there exists at least one T-periodic solution.

These results extend the classical criteria for the Liénard equation to cases with non-monotonic damping and bounded external forcing. The use of the modified energy-logarithmic Lyapunov function allows for sharper bounds on the parameters a and b compared to standard quadratic forms.

4. Extensions to Higher Order Systems

The methodology described above can be generalized to systems of the form $\dot{x}_i = f_i(t, x_1, \ldots, x_n)$. If there exists a positive definite function $V(t, x)$ such that its derivative $\dot{V}$ is negative outside a certain sphere, the system remains bounded. In our case, the specific construction of V using the integral of the damping function $\Phi(x)$ provides a robust mechanism for proving stability in the presence of significant nonlinearities.

summarizes the stability regions for different values of the parameters a and c, indicating the transition from periodic behavior to global asymptotic stability.

References

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Note: Figure translations are in progress. See original paper for figures.

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