EXHAUSTIVE SOLUTION IN THE CLASSES W_2^{α} AND $C^{(n,\alpha)}$ OF THE LOCALIZATION PROBLEM FOR FOURIER SERIES IN FUNDAMENTAL SYSTEMS OF FUNCTIONS OF THE LAPLACE OPERATOR

MATHEMATICS

1967

Abstract

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UDC 517.521.5

MATHEMATICS

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EXHAUSTIVE SOLUTION IN THE CLASSES W_2^α AND $C^{(n,\alpha)}$ OF THE LOCALIZATION PROBLEM FOR FOURIER SERIES IN FUNDAMENTAL SYSTEMS OF FUNCTIONS OF THE LAPLACE OPERATOR

(Presented by Academician A. N. Tikhonov on January 6, 1967)

The paper studies Fourier series with respect to arbitrary fundamental systems of functions of the Laplace operator in an arbitrary N-dimensional domain g*. Particular cases of the Fourier series under study are Fourier series in eigenfunctions of any of the three homogeneous boundary-value problems and multiple trigonometric Fourier series. For the indicated Fourier series an exhaustive solution, in the classes W_2^α and $C^{(n,\alpha)}$, of the localization problem is given. All the results obtained are new even for multiple trigonometric Fourier series.

1°. Formulation of the results. Divergence of the Fourier series of a function $f(x)$ at some interior point x_0 of the domain g may be due to three causes: 1) insufficient smoothness of the function $f(x)$ in a small neighborhood of the point x_0 (i.e., absence of local smoothness); 2) insufficient smoothness of $f(x)$ at points of the domain lying outside some neighborhood of the point x_0 (i.e., absence of smoothness conditions ensuring the localization principle for the Fourier series under consideration); 3) the influence of boundary conditions, i.e., the circumstance that the function $f(x)$, although sufficiently smooth in the entire closed domain $\bar{g}$, is such that this function itself or at least one of its repeated Laplacians up to some order l (related to the number N of dimensions of the domain) does not satisfy the boundary condition to which the functions $u_n(x)$ are subject**.

To eliminate the influence of local smoothness and the influence of boundary conditions it is natural to consider a function equal to zero in some neighborhood of the point x_0 under consideration and in some boundary strip of the domain g.

We have proved the following two theorems.

Theorem 1 (on conditions not ensuring localization). *Let g be an arbitrary N-dimensional domain, $N \geq 2$; let x_0 be any fixed*

* In accordance with the work ([1]), we call any complete orthonormalized system $\{u_n(x)\}$ in an arbitrary N-dimensional domain g a **fundamental system of functions of the Laplace operator** if each function $u_n(x)$ belongs in the open domain g to the class $C^{(2)}$ and, for some nonnegative number λ_n, satisfies in g the equation $\Delta u_n + \lambda_n u_n = 0$. In doing so we completely refrain from introducing boundary conditions in any form.

** In my paper ([2]) (see pp. 177-180) an example is constructed of a function $f(x)$ satisfying the following conditions: 1) $f(x)$ is arbitrarily smooth in a closed N-dimensional ball Ω; 2) at least one (and arbitrary!) of the Laplacians of $f(x)$ up to some order l does not satisfy on the boundary of the ball Ω the boundary condition to which the eigenfunctions of the ball are subject (here $l = [(N-3)/4]$ for $N \geq 3$ and for eigenfunctions of the first boundary-value problem, and $l = [(N-5)/4]$ for $N \geq 5$ and for eigenfunctions of the second boundary-value problem); 3) the Fourier series of the function $f(x)$ in the eigenfunctions of the ball Ω diverges at the center x_0 of this ball (for any order of arrangement of its terms!).

an interior point of the domain g; α is any fixed number satisfying the inequalities

$$0 < \alpha < \frac{1}{2} \quad \text{for even } N, \qquad 0 < \alpha < 1 \quad \text{for odd } N. \tag{1}$$

Then there exists a function $f(x)$ satisfying the following requirements: 1) $f(x)$ is equal to zero in some neighborhood of the point x_0 and in some boundary strip of the domain g; 2) $f(x)$ in the closed domain $\bar{g}$ belongs to the class $C^{([N/2]-1,\alpha)}$, where α satisfies the inequalities (1); 3) the Fourier series of the function with respect to the fundamental system of functions of the Laplace operator in the domain g diverges at the point x_0.*

Theorem 1 says that, for convergence of the Fourier series of the function $f(x)$ at the point x_0, it is not enough that this function belong to the class $C^{([N/2]-1,\alpha)}$ with Hölder exponent α satisfying the inequalities (1) (despite the fact that the function $f(x)$ is arbitrarily smooth in a neighborhood of the point x_0 and is equal to zero in a boundary strip of the domain).

Corollary. For any real $\alpha < (N-1)/2$ there exists a function $f(x)$ satisfying requirements 1) and 3) of Theorem 1 and belonging to the class $W_2^\alpha(g)$.

Naturally the question arises whether the localization principle will be valid if the value of the exponent α is increased. The answer to this question is given by the following assertion.

Theorem 2. Let g be an arbitrary N-dimensional domain; $N \geq 2$; and let α be a number satisfying the relations

$$\alpha > \frac{1}{2} \quad \text{for even } N, \qquad \alpha = 1 \quad \text{for odd } N, \tag{2}$$

and let $f(x)$ be an arbitrary function satisfying two requirements: 1) $f(x)$ is equal to zero in some boundary strip of the domain g; 2) $f(x)$ in the closed domain $\bar{g}$ belongs to the class $C^{([N/2]-1,\alpha)}$, where α satisfies the relations (2).

Then the Fourier series of the function $f(x)$ with respect to an arbitrary fundamental system of functions of the Laplace operator in the domain g converges to the function $f(t)$ everywhere inside the domain g; moreover, the convergence of this series is uniform in every strictly interior subdomain g' of the domain g.

Thus, if the Hölder exponent α satisfies not the inequalities (1) but the relations (2), then belonging to the class $C^{([N/2]-1,\alpha)}$ ensures not only the localization principle for the Fourier series, but is also a local smoothness condition sufficient for convergence of the Fourier series.

Remark. In Theorem 2, requirement 2) imposed on the function $f(x)$ may be somewhat weakened. If, instead of condition 2), one requires that the function $f(x)$ belong in the whole domain g to the class $W_2^{(N-1)/2}(g)$, and in some neighborhood Ω of the point x_0 under consideration belong to the class $W_p^{(N-1)/2}(\Omega)$, where $p > 2N/(N-1)$, then Theorem 2 will assert convergence of the Fourier series of the function $f(x)$ everywhere inside Ω, uniformly in every strictly interior subdomain Ω' of the domain Ω.

We arrive at the following **conclusions**:

1) In the classes W_2^{α}, final conditions for localization have been obtained: for $\alpha \geq (N-1)/2$ the localization principle is valid, while for $\alpha < (N-1)/2$ localization does not take place.

2) In the classes $C^{(n,\alpha)}$, final conditions for localization and convergence have been obtained in the case of an **odd** number N of dimensions: membership of $f(x)$ in the class $C^{([N/2]-1,\alpha)}$ with $\alpha < 1$ is insufficient even for localization, while membership of $f(x)$ in the same class with $\alpha = 1$ is sufficient not only for localization but also for convergence.

3) In the case of an **even** number N of dimensions, membership in the class $C^{(N/2-1,\alpha)}$ does not ensure localization for $\alpha < \frac{1}{2}$ and ensures both local-

* The order of summation of the terms of the Fourier series is considered to coincide with the order of increase of the fundamental numbers λ_n.

localization and convergence for $\alpha > 1/2$. (For $\alpha = 1/2$ and under the condition that $f(x)$ is equal to zero in a neighborhood Ω of the point x_0, we were able to prove that the partial sums of the Fourier series of $f(x)$ are uniformly bounded in every strictly interior subdomain Ω' of the neighborhood Ω.)

2°. Let us pass to the main ideas of the proof of Theorems 1 and 2. We note at once that the proofs required a number of rather delicate and nontrivial arguments. The positive Theorem 2 was easier for the author, since in proving this theorem one may take as a basis the work ([2]), and, in comparison with that work, certain serious difficulties have to be overcome only in the four-dimensional case. The negative Theorem 1 required quite delicate technique. The proof of this theorem essentially uses the following lemma.

Lemma. *Let an arbitrary positive number ε, an arbitrary point x_0 of the open domain g, and a certain closed domain E, contained in the open domain g, not containing the point x_0, and possessing the property that there exists a ball, lying entirely in g, of some radius R with center at the point x_0, whose intersection with E is a set containing interior points, be fixed.*

Then the sequence

$$F_n(x_0) = \int_E \left| \sum_{k=1}^{n} \frac{u_k(x_0)u_k(x)}{\lambda_k^{(N-1)/4-\varepsilon}} \right| dy \tag{3}$$

is unbounded.

The proof of the lemma is carried out by contradiction. Assuming the boundedness of the sequence (3), we manage to prove the existence of such a positive constant β that

$$\int_E \left| \sum_{k=1}^{n} u_k(x_0)u_k(y) \right| dy > \beta \lambda_n^{(N-1)/4}, \tag{4}$$

whereas this contradicts the estimate, following from the boundedness of the sequence (3),

$$\int_E \left| \sum_{k=1}^{n} u_k(x_0)u_k(y) \right| dy = O\left(\lambda_n^{(N-1)/4-\varepsilon}\right).$$

Naturally, we cannot here even describe those nontrivial arguments which lead to the proof of inequality (4). After the lemma has been proved, in order to prove Theorem 1, use is essentially made of the properties of kernels of fractional order, constructed by the author in ([3]), and of Banach theorems of resonance type. This allows us to establish the existence of a function $h(y)$, continuous in the domain E, such that the Fourier series of the function

$$F(x) = \int_E K_{(N-1)/4-\varepsilon}(x, y)h(y)\, dy,$$

where $K_{(N-1)/4-\varepsilon}(x, y)$ is the kernel of fractional order, diverges at the point x_0. With the help of the properties of kernels of fractional order, it is possible to study the differential properties of the function $F(x)$ and to construct, on the basis of this function, the function $f(x)$ which constitutes the content of Theorem 1.

The author expresses his gratitude to A. N. Tikhonov, D. E. Menshov, and P. L. Ulyanov; conversations and discussions of the results with them led to the appearance of the present work in the exhaustive form presented above.

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Received
27 XII 1966

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Note: Figure translations are in progress. See original paper for figures.