ON A SINGULAR PROBLEM WITH CONDITIONS OF F. I. FRANKL AND F. TRICOMI

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Abstract

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MATHEMATICS

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ON A SINGULAR PROBLEM WITH CONDITIONS OF F. I. FRANKL AND F. TRICOMI

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1. Consider the equation

$$S(u) \equiv u_{xx} + \operatorname{sgn} y\, u_{yy} + \frac{2q}{x} u_x = 0, \qquad q > 1 \tag{S}$$

in the domain D, bounded by: 1) the arc AB of the circle $x^2 + y^2 = 1$, $A(1;0)$, $B(0;1)$; 2) the segment OB of the axis OY; 3) the segment OC of the characteristic $x + y = 0$, $C(c;-c)$, $c \geq 1/2$; 4) the chordal arc AC, which either coincides completely with the segment CI $(I(2c;0))$ of the characteristic $x - y = 2c$, or is situated between this characteristic and the segment AE $(E(1/2;-1/2))$ of the characteristic $x - y = 1$. The arc AC is such that any characteristic can intersect it in no more than one point. Let D_1 and D_2 be, respectively, the subdomains of ellipticity and hyperbolicity of equation (S). The magnitude of the arc s along the boundary of the domain D will be measured from the point A counterclockwise. Let l be the length of AC, and TR a segment of the arc AB, whose length is equal to the length of AC and whose endpoints T and R have respectively the arc coordinates $\varepsilon > 0$, $\varepsilon + l < \pi/2$.

Singular problem $\widetilde{A}_1$. In the domain D, find a function $u(x,y)$ possessing the properties: $S(u) = 0$ in $D_1 \cup D_2$, $u \in C(\overline{D})$, $u \in C^1(D \setminus OA)$,

$$u_y(x; 0-0) = u_y(x; 0+0)$$

(the "gluing" condition), and satisfying the following boundary conditions:

$$\begin{array}{llll} 1. & u\big|_{\overline{AT}} = \gamma_1(s). & 3. & u\big|_{\overline{OC}} = \psi(x). \\ 2. & u\big|_{\overline{RB}} = \gamma_2(s). & 4. & u(\varepsilon+\mu) - u(-\mu) = g(\mu), \quad 0 \leq \mu \leq l. \end{array} \tag{I}$$

Here $\gamma_1, \gamma_2, \psi, g$ are prescribed functions. Condition 4 defines a curvilinear jump of compactification [1–3].

A **generalized solution** of problem $\widetilde{A}_1$ of class $\widetilde{\mathcal{G}}_2$ (of class $\widetilde{\mathcal{G}}_2^0$) will mean a function $u(x,y)$, defined in D and satisfying the conditions: 1) $S(u)=0$ in D_1; 2) $u \in C^2(D_1)$; 3) $u \in C(\overline{D})$; 4) u is a generalized solution of equation (S) of class $\widetilde{\mathcal{G}}_2$ (class $\widetilde{\mathcal{G}}_2^0$) in the domain D_2 (such classes of solutions are introduced below); 5) for any x, $0<x<1$, there exists

$$\lim_{y\to 0} u_y = v(x),$$

where $v(x)$ is absolutely integrable on $0 \le x \le 1$ and satisfies the Hölder condition on $0<x<1$; 6) u satisfies the boundary conditions (I).

2. Problem NE. In the domain D_1, find a bounded solution of equation (S) from the data:

$$\lim_{y\to 0} u_y = v(x), \quad 0<x<1; \qquad u\big|_{AB} = \varphi(s). \tag{II}$$

Without loss of generality one may assume

$$\gamma_1(0) = \gamma_2(\pi/2) = \psi(0) = 0.$$

In the domain D_1, the generalized solution of problem $\widetilde{A}_1$ of class $\widetilde{\mathcal{G}}_2$ is determined by the formula [(4)]

$$u(x_0,y_0) = -\int_0^1 G_0(t,0;x_0,y_0)v(t)\,dt - \int_0^{\pi/2} \frac{\partial G_0}{\partial n}\,\varphi(s)\,ds, \tag{1}$$

where $G_0(x,y;x_0,y_0)$ is the Green function of problem NE.

Lemma 1. Let $u(x,y)$ be a generalized solution of problem $\widetilde{A}_1$ of class $\widetilde{\mathcal{G}}_2$. Then $u(x,y)$ cannot attain in $\bar{D}_1$ a greatest positive (least negative) value on the interval $x=0,\ 0<y<1$, greater (smaller) than the values of $u(x,y)$ on $\overline{AB} \cup \overline{AO}$.

3. Problem $\widetilde{\mathcal{G}}_2$. In the domain D_2, find a solution $u(x,y)$ of equation (S) from the data

$$\lim_{y\to 0} u_y = \nu(x), \quad 0<x<1; \qquad u\big|_{OC} = \psi(x). \tag{III}$$

By the change of variables $\xi = y+x,\ \eta = y-x$ in the domain D_2, equation (S) is transformed to the form

$$S_0(u) \equiv u_{\xi\eta} - \frac{q}{\xi-\eta}(u_\xi - u_\eta) = 0.$$

The domain D_2 goes over into the triangle $O'A'C'$ (the domain Δ). The boundary conditions (III) take the form

$$\lim_{(\xi,\eta)\to(x;-x)}(u_\xi+u_\eta)=\nu(x),\quad 0<x<1;\qquad u(0;\eta)=\psi(-\eta/2)=\psi_1(\eta).$$

We shall call the function $u(\xi,\eta)$ a **regular solution** of equation (S_0) in Δ (of equation (S) in D_2) if $S(u)=0$ in Δ, $u\in C^2(\Delta)$, $u\in C(\bar{\Delta})$, $u\in C^1(\bar{\Delta}\setminus(O'\cup A'))$.

Lemma 2. If the function $\nu(\xi)$ is continuously differentiable on $0<\xi<1$ and absolutely integrable on $0\le\xi\le1$, the function $\psi_1(\eta)$ is continuous on $-2c\le\eta\le0$, twice continuously differentiable on $(-2c,0)$, and $\psi_1(\eta)=(-\eta)^\alpha\widetilde{\psi}(\eta)$ $(0<\alpha\le1,\ \widetilde{\psi}(\eta)$ bounded), then the regular solution $u(\xi_0,\eta_0)$ of equation (S_0) is determined by the formula

$$u(\xi_0,\eta_0)=\int_0^{\xi_0}H(\xi,\xi_0,\eta_0)\nu(\xi)\,d\xi-\int_{\eta_0}^0\left[\frac{\partial A(0,\eta;\xi_0,\eta_0)}{\partial\eta}-\frac{q}{\eta}A(0,\eta;\xi,\eta_0)\right]\times$$
$$\times\,\psi_1(\eta)\,d\eta+\left(\frac{\xi_0}{\xi_0-\eta_0}\right)^q\psi_1(-\xi_0)+\left(\frac{-\eta_0}{\xi_0-\eta_0}\right)^q\psi_1(\eta_0). \tag{2}$$

Here $H(\xi,\xi_0,\eta_0)$ is the value for $\xi=-\eta$ of the Riemann function; $A(\xi,\eta;\xi_0,\eta_0)$ is the Riemann-Hadamard function of problem $C\mathcal{G}$ ([4]). The proof is analogous to ([5]), Theorem 6.

By a **generalized solution of class** $\widetilde{\mathcal{G}}_2$ of equation (S) in the domain D_2 we shall mean the function defined by formula (2), in which $\nu(\xi)$ is continuous on $0<\xi<1$, $\psi_1(\eta)$ is continuous on $-2c\le\eta\le0$, and

$$\psi_1(\eta)=(-\eta)^\alpha\widetilde{\psi}(\eta)\quad(0<\alpha<1,\ \widetilde{\psi}(\eta)\text{ bounded}).$$

If, in addition, $\psi'(0)=0$, $\nu(\xi)$ and $\psi_1'(\eta)$ satisfy a Hölder condition respectively on $0<\xi<1$ and $-2c\le\eta\le0$, then the function $u(\xi_0,\eta_0)$ will be regarded as belonging to the class $\widetilde{\mathcal{G}}_2^0$ of generalized solutions of equation (S) in D_2.

Lemma 3. Every generalized solution of class $\widetilde{\mathcal{G}}_2$ of equation (S) in the domain D_2 can be represented as the limit of a sequence, uniformly convergent in $\bar{D}_2$, of regular solutions of equation (S) in D_2.

Lemma 4. Let $u(\xi,\eta)$ be a regular solution of equation (S_0), with $u(0,\eta)=0$. Then the maximum of $u(\xi,\eta)$ in $\bar{\Delta}$, if it is positive, is attained only on the segment $O'A'$.

The proof follows from $q/(\xi-\eta)>0$, $(q-q^2)/(\xi-\eta)<0$ in Δ.

Lemma 5. Let $u(\xi,\eta)$ be a generalized solution of class $\widetilde{\mathcal{G}}_2$ of equation (S) in D_2, with $u(0,\eta)=0$. Then the maximum of $u(\xi,\eta)$ in $\bar{\Delta}$, if it is positive, is attained on the segment $O'A'$.

The proof follows from Lemmas 3 and 4.

4. Theorem 1. There can exist no more than one generalized solution of problem $\widetilde{A}_1$ of class $\widetilde{\mathcal{G}}_2$.

Suppose the contrary. Then there exists a generalized solution $u(x,y)$ of problem $\widetilde{A}_1$ of class $\widetilde{\mathcal{G}}_2$, taking positive values, satisfying

satisfying homogeneous boundary conditions. Suppose the maximum of $u(x,y)$ in $\overline{D}$ is attained in $\overline{D}_2$. Then, by Lemma 5, it is attained at an interior point of the segment OA, which contradicts the gluing condition. Assuming that the maximum of $u(x,y)$ in $\overline{D}$ is attained in $\overline{D}_1$, by virtue of the preceding, Lemma 1 and the density jump condition we obtain that it is attained in $\overline{D}_2$, which is impossible.

5. Solution of problem $\widetilde{A}_1$. Let $u(x,y)$ be a generalized solution of problem $\widetilde{A}_1$ of the class $\mathcal{G}_2^0$. Taking into account (1), (2) and the gluing condition, we obtain an integral equation of the first kind with unknown function $\nu(x)$. Differentiating and transforming this equation, we arrive at the singular integral equation

$$\nu(x)+\frac{1}{\pi}\int_0^1\left[\frac{1}{t-x}-\frac{1}{1-tx}+\frac{\mathcal{L}(t,x)}{t+x}\right]\nu(t)\,dt=\int_0^1 R(t,x)\nu'(t)\,dt+f(x), \quad (3)$$

where $\mathcal{L}(t,x)$ is a homogeneous function, not depending, like $R(t,x)$, on the boundary conditions. The kernel $R(t,x)$ is continuous everywhere in the square $0\le t,x\le 1$, with the exception of the point $(1;1)$, at which it has a singularity of arbitrarily small order. The function $f(x)$ depends on $\gamma_1(s)$, $\gamma_2(s)$, $\psi(x)$, $g(\mu)$, $\nu(x)$.

Lemma 6. In equation (3) the function $f(x)$ can be represented in the form

$$f(x)=f_0(x)+\int_0^1 Z(t,x)\nu(t)\,dt, \quad (4)$$

where $f_0(x)$ does not depend on $\nu(x)$, and the kernel $Z(t,x)$ is continuous in the square $0\le t,x\le 1$.

Taking Lemma 6 into account, we write equation (3) in the form

$$\nu(x) + \frac{1}{\pi}\int_0^1 \left[\frac{1}{t-x} - \frac{1}{1-tx} + \frac{\mathcal{L}(t,x)}{t+x}\right]\nu(t)\,dt = \int_0^1 R_0(t,x)\nu(t)\,dt + f_0(x), \tag{5}$$

where $R_0(t,x) = R(t,x) + Z(t,x)$.

Lemma 7. If the function $\gamma_1(s)$ has a derivative of second order at least in a neighborhood of the point $s = 0$, then the function $f_0(x)$ satisfies the Hölder condition on $0 \le x \le 1$.

Applying the method of work (6), we reduce equation (5) to the equivalent Fredholm equation

$$\varphi(\xi) = \int_0^\infty R_{00}(\eta,\xi)\varphi(\eta)\,d\eta + f_{00}(\xi), \tag{6}$$

$$\varphi(\xi) = e^{-\xi}\nu(e^{-2\xi}), \qquad x = e^{-2\xi}, \qquad t = e^{-2\eta}. \tag{7}$$

The function $f_{00}(\xi) \equiv 0$ if $\gamma_1(s) = \gamma_2(s) = \psi(x) = g(\mu) \equiv 0$. From Theorem 1 follows the uniqueness and existence of a solution of equation (6) in the class of functions satisfying the Hölder condition and absolutely integrable with some power $p > 1$ on $0 < \xi < +\infty$. Having found $\varphi(\xi)$ from (6), we find $\nu(x)$ from (7). By formula (2) we find $u(x,y)$ in $\overline{D}_2$, and, consequently, also $u|_{\overline{AC}}$. With the help of the density jump condition we find $u|_{\overline{TR}}$, and then by formula (1) determine $u(x,y)$ in $\overline{D}_1$. Thus, the following is valid.

Theorem 2. If the function $\gamma_1(s)$ has a derivative of second order at least in a neighborhood of the point $s = 0$, then there exists a unique generalized solution of problem $\widetilde{A}_1$ of the class $\mathcal{G}_2^0$.

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Note: Figure translations are in progress. See original paper for figures.