ANALYTIC CAPACITY AND THE CAUCHY INTEGRAL

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ANALYTIC CAPACITY AND THE CAUCHY INTEGRAL

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The paper gives an estimate of the Cauchy integral over a contour in terms of the analytic capacity $\gamma(E)$ ([2,3]) of the set E of singular points of the integrand.

Theorem 1. Let E be a closed set in the unit disk $|z| \leq 1$; let $f(z)$ be a function analytic inside the disk outside E, $|f(z)| \leq 1$, and let $f(z)$ be continuous on the unit circle. Then

$$\left| \int_{|\zeta|=1} f(\zeta)\, d\zeta \right| \leq c\gamma(E),$$

where c is an absolute constant.

This problem, which arose in approximation theory, was posed by A. G. Vitushkin ([1]), §4.

Here, with the aid of the estimate obtained, we prove the semiadditivity of analytic capacity for two sets separated by an analytic curve (Corollary 4).

Lemma 1. Let E be a bounded closed set; let $f(z)$ be analytic outside E; $f(\infty) = 0$; $\operatorname{Im} f(z) \leq a$, $a > 0$. Then $|f'(\infty)| \leq 2a\gamma(E)$.

The lemma is proved by the transformation

$$\tilde{f}(z) = f(z)[f(z) - 2ia]^{-1}.$$

Lemma 2. Let E be a bounded closed set; let the point a belong to the component of the complement of E containing infinity; let $\rho = \rho(a, E)$ be the distance from a to E; let $f(z)$ be analytic outside E, $|f(z)| \leq 1$. Then

$$\rho[\rho - \gamma(E)]|f'(a)| \leq 2\gamma(E).$$

The lemma follows from the equality

$$f'(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\, d\zeta}{(\zeta - a)^2} = \frac{1}{2\pi i} \int_\Gamma \left[\frac{f(\zeta) - f(a)}{(\zeta - a)^2} - \frac{f(a)}{\zeta - a} \right] d\zeta,$$

where Γ is a closed rectifiable curve separating E from infinity and from the point a, but not separating a from ∞; moreover one may assume that $\rho(a,\Gamma)$ is arbitrarily close to ρ.

Remark. If one further assumes that $f(a)=0$, then we obtain

$$\rho[\rho-\gamma(E)]|f'(a)| \le \gamma(E).$$

From this remark, by the transformation

$$\varphi(z)=\tilde{f}(1/z),$$

one immediately derives

Lemma 3. Let a closed set E lie in the annulus $r \le |z| \le 1$; let E^* be the set symmetric to E with respect to the unit circle. Then

$$r^2\gamma(E^*) \le 2\gamma(E).$$

Theorem 2. Let E be a closed set in the unit disk $|z|\le 1$; let $0\in E$; let E^* be symmetric to E with respect to the unit circle. Then

$$\gamma(E\cup E^*) \le 10[\gamma(E)+\gamma(E^*)].$$

Proof. Let E_δ, E^*_δ be the δ-neighborhoods of the sets E and E^*. Obviously, as $\delta\to 0$, $\gamma(E_\delta)\to\gamma(E)$, $\gamma(E^*_\delta)\to\gamma(E^*)$; therefore it is enough

to show that $\gamma(E\cup E^*) \le 10[\gamma(E_\delta)+\gamma(E^*_\delta)]$ for every $\delta>0$. To this end we shall prove that $|f'(\infty)| \le 10[\gamma(E_\delta)+\gamma(E^*_\delta)]$ for any function $f(z)$, analytic outside $E\cup E^*$, $|f(z)|\le 1$, $f(\infty)=0$. We note that then $f(z)$ may be assumed continuous in the whole plane and analytic outside $E_\delta\cup E^*_\delta$. Let $z^*=1/\overline{z}$.

Denote by $\varphi(z)$ that one of the two functions $f(z)+\overline{f(z^*)}$, $f(z)-\overline{f(z^*)}$, for which the modulus of the derivative at infinity is not less than $|f'(\infty)|$; then $|\varphi(z)|\le 2$, $\varphi(z)$ is also analytic outside $E_\delta\cup E^*_\delta$ and continuous everywhere, while on the unit circle $\varphi(z)$ is either purely real or purely imaginary, depending on the choice.

Consider the functions

$$\varphi_1(z)=\begin{cases}\varphi(z)-\dfrac{1}{2\pi i}\displaystyle\int_{|\zeta|=1}\frac{\varphi(\zeta)\,d\zeta}{\zeta-z}, & \text{for } |z|<1,\\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{|\zeta|=1}\frac{\varphi(\zeta)\,d\zeta}{\zeta-z}, & \text{for } |z|>1;\end{cases}$$

$$\varphi_2(z)=\begin{cases}\varphi(z)-\dfrac{1}{2\pi i}\displaystyle\int_{|\zeta|=1}\frac{\varphi(\zeta)\,d\zeta}{\zeta-z}, & \text{for } |z|>1,\\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{|\zeta|=1}\frac{\varphi(\zeta)\,d\zeta}{\zeta-z}, & \text{for } |z|<1.\end{cases}$$

It is easy to see that $\varphi_1(z)$ and $\varphi_2(z)$ can be extended by continuity at the points of the unit circle not belonging to $E_\delta \cup E_\delta^*$; then $\varphi_1(z)$ and $\varphi_2(z)$ are analytic respectively outside E_δ and E_δ^*, $\varphi_1 + \varphi_2 = \varphi$, $\varphi_1(\infty) = 0$, $|\varphi_2(\infty)| \leq 1$.

It is well known that

$$\left|\operatorname{Im}\left[\int_{|\zeta|=1} \frac{\tau(\zeta)\, d\zeta}{\zeta - z}\right]\right| \leq 2\pi \max_{|\zeta|=1} |\tau(\zeta)|$$

for any real continuous function $\tau(\zeta)$. Therefore, depending on the choice of $\varphi(z)$, either $\operatorname{Im}[\varphi_1(z)] \leq 4$, $\operatorname{Im}[\varphi_2(z)] \leq 4$, or $\operatorname{Re}[\varphi_1(z)] \leq 4$, $\operatorname{Re}[\varphi_2(z)] \leq 4$; but then, by Lemma 1, we obtain

$$|f'(\infty)| \leq |\varphi'(\infty)| \leq |\varphi_1'(\infty)| + |\varphi_2'(\infty)| \leq 10[\gamma(E_\delta) + \gamma(E_\delta^*)].$$

The theorem is proved.

Theorem 3. Let E be a closed set in the annulus $r \leq |z| \leq 1$; let $f(z)$ be a function analytic inside the disk outside E, continuous in the whole disk; $|f(z)| \leq 1$. Then

$$\left|\int_{|\zeta|=1} f(\zeta)\, d\zeta\right| \leq c\, \frac{1}{r^2}\, \gamma(E),$$

where c is an absolute constant. It may be assumed that $f(z)$ is continuous only on the unit circle.

Proof. Without loss of generality, one may assume $f(z)$ smooth at the points of the unit circle not belonging to the set E. If E separates 0 from infinity, then $\gamma(E) \geq r$, and the assertion of the theorem is obvious. Consider the functions

$$\varphi_{+,-}(z) = \begin{cases} f(z), & \text{for } |z| < 1, \\ \pm\overline{f(1/\overline{z})}, & \text{for } |z| > 1, \end{cases}$$

then the functions

$$\psi_{+,-}(z) = \varphi_{+,-}(z) - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta) \pm \overline{f(\zeta)}}{\zeta - z}\, d\zeta,$$

extended by continuity at the points of the unit circle not belonging to E, will be analytic outside $E \cup E^*$, and moreover (analogously to the theorem 2) $\operatorname{Re}[\psi_+(z)] \leqslant 3$, $\operatorname{Im}[\psi_-(z)] \leqslant 3$. Then, by Lemma 1 and Theorem 2, we have

$$|\psi'_{+,-}(\infty)| \leqslant 6[\gamma(E \cup E^*)] \leqslant 60[\gamma(E) + \gamma(E^*)].$$

Further,

$$\left|\int_{|\zeta|=1} f(\zeta)d\zeta\right| \leqslant \frac{1}{2}|\psi'_+(\infty) + \psi'_-(\infty)| \leqslant 60[\gamma(E) + \gamma(E^*)],$$

whence, by Lemma 3, we obtain the assertion of the theorem.

Proof of Theorem 1. Without loss of generality, one may assume that the set E is bounded by a finite number of rectifiable closed curves, and that $f(z)$ is analytic inside the unit disk outside E and is smooth everywhere. Denote by $\beta(z)$ a real-valued infinitely differentiable function in the unit disk with the properties: $\beta(z) = 1$ for $1 \geqslant |z| \geqslant {}^2/_3$; $\beta(z) = 0$ for $|z| \leqslant {}^1/_3$; $|\beta(z)| \leqslant 1$, $|\partial\beta/\partial\bar{z}| \leqslant 4$. Then

$$\beta f + (1-\beta)f = f.$$

By the Cauchy-Green formula, one may represent $\beta f = f_1 + \varphi_1$, $(1-\beta)f = f_2 + \varphi_2$, where $f_1(z)$ is analytic in the disk outside that part of E which lies in the annulus ${}^1/_3 \leqslant |z| \leqslant 1$; $f_2(z)$ is analytic in the disk outside that part of E which lies in the disk $|z| \leqslant {}^2/_3$; $|f_1(z)| \leqslant c_1$; $|f_2(z)| \leqslant c_1$; $\varphi_1(z) + \varphi_2(z) = 0$. Then

$$\left|\int_{|\zeta|=1} f(\zeta)d\zeta\right| \leqslant \left|\int_{|\zeta|=1} f_1(\zeta)d\zeta\right| + \left|\int_{|\zeta|=1} f_2(\zeta)d\zeta\right|,$$

and the assertion of the theorem follows at once from Theorem 3 (with $r = {}^1/_3$) and the obvious fact that the estimate of the integral holds if the set lies in the disk $|z| \leqslant {}^2/_3$.

Corollary 1. If the sets E_i lie, respectively, in nonintersecting disks K_i, then

$$\gamma\left(\bigcup_i E_i\right) \leqslant c\sum_i \gamma(E_i).$$

Corollary 2. Let E be a closed set situated in the annulus ${}^1/_2 \leqslant |z| \leqslant 1$; $f(z)$ a continuous function, analytic in this annulus outside the set E; $|f(z)| \leqslant 1$. Then

$$\left|\int_{|\zeta|=1} f(\zeta)d\zeta - \int_{|\zeta|={}^1/_2} f(\zeta)d\zeta\right| \leqslant c\gamma(E).$$

This corollary can be obtained in the same way as the proof of Theorem 1. Corollary 2 makes it possible to prove

Corollary 3. Let E be a bounded closed set; let $a \in E$; $K(n)$ be the annulus $^1/_{2^n} \geqslant |z-a| \geqslant {}^1/_{2^{n+1}}$. In order that there exist a function $f(z)$, admitting uniform approximation on E by rational fractions and such that $f(a) = 1,\ f(z) < 1$ for $z \neq a$, it is necessary and sufficient that the series

$$\sum_{n=0}^{\infty} 2^n \gamma[K(n) \setminus E]$$

converge.

Theorem 4. Let the domain G be bounded by a simple closed analytic curve Γ; let the closed set $E \subset \overline{G}$, and let $f(z)$ be a continuous function, analytic in $G \setminus E$; $|f(z)| \leqslant 1$. Then

$$\left| \int_\Gamma f(\zeta) d\zeta \right| \leqslant c(G)\gamma(E),$$

where $c(G)$ is a constant depending only on the domain G.

Without loss of generality, one may assume that $E \subset G$ and that its boundary ∂E consists of a finite number of closed rectifiable curves. In this situation, for the proof of Theorem 4 we shall use dual formulations for the extremal problems arising in the definition of ana-

analytic capacity and of the maximum of the Cauchy integral (see, for example, $(^4)$). Let $A(E)$ be the set of functions $f(z)$, analytic in CE, such that $|f(z)| \leq 1$, $f(\infty) = 0$; let $H_1(E)$ be the space of type H_1 of functions analytic in CE and equal to 0 at infinity; let $A(G \setminus E)$ be the set of functions analytic in $G \setminus E$, $|f(z)| \leq 1$; and let $H_1(G \setminus E)$ be the space of type H_1 of functions analytic in $G \setminus E$. Then

$$\gamma(E) = \max_{f \in A(E)} \left| \int_{\partial E} f(\zeta)\, d\zeta \right| = \min_{g \in H_1(E)} \int_{\partial E} |1 - g|\, ds,$$

$$I(G, E) = \max_{f \in A(G \setminus E)} \left| \int_\Gamma f(\zeta)\, d\zeta \right| = \min_{g \in H_1(G \setminus E)} \left[\int_\Gamma |1 - g|\, ds + \int_{\partial E} |g|\, ds \right].$$

If G satisfies the condition of the theorem, then there exists a conformal mapping $w(z)$ of the domain G onto the unit disk K, which extends to a conformal mapping of some fixed neighborhood $U(G)$ onto a neighborhood $U(K)$, and moreover

$$0 < m \leq |w'(z)| \leq M.$$

Let $E_1 \subset K$ be the image of E under the mapping $w(z)$, and let $g(w) \in H_1(K \setminus E_1)$ be a function such that

$$I(K, E_1) = \int_{|w|=1} |1 - g(w)|\, ds + \int_{\partial E_1} |g(w)|\, ds.$$

Then

$$I(G, E) \leq \int_{\Gamma} |1 - g(w(z))|\, ds + \int_{\partial E} |g(w(z))|\, ds \leq \frac{1}{m} I(K, E_1) \leq \frac{1}{m} c\gamma(E_1).$$

It remains to estimate $\gamma(E_1)$ in terms of $\gamma(E)$; this is done with the aid of the inverse mapping $z(w)$, on the basis of the fact that

$$\gamma(E) \geq c(G) I(U(G), E) \geq c(G) \frac{1}{M} I(U(K), E_1) \geq \frac{1}{M} c(G) \gamma(E_1).$$

Corollary 4. If two sets E_1 and E_2 are separated by an analytic curve Γ, then

$$\gamma(E_1 \cup E_2) \leq c(\Gamma)\, [\gamma(E_1) + \gamma(E_2)].$$

Theorem 5. If a bounded closed set E is such that its interior boundary (i.e., boundary points which are not boundary points for components of the complement) lies on an analytic curve, then every function continuous on E and analytic at the interior points of E can be uniformly approximated on E by rational functions of z.

The proof follows from Theorems 3 and 7 of (^{5}) and Theorem 4.

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