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Representations of the solution to a system of linear differential equations in the neighborhood of an irregular singular point

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

APRIL 1967, VOLUME III, NO. 4

ON REPRESENTATIONS OF SOLUTIONS FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS IN THE NEIGHBORHOOD OF AN IRREGULAR SINGULAR POINT

Consider the system of linear differential equations

$$\frac{dy}{dx} = A(x)y \tag{0}$$

where x is a complex variable, y is an n-dimensional vector, and $A(x)$ is an $n \times n$ matrix whose elements are analytic in the neighborhood of the point $x = \infty$ and can be represented by convergent series

$$A(x) = \sum_{k=0}^{\infty} A_k x^{-k} \tag{1}$$

The point $x = \infty$ is, in general, an irregular singular point for system (0).

As is well known, if the eigenvalues of the matrix A_0 are distinct, then system (0) has a formal fundamental solution of the form

$$Y(x) = P(x)x^W \exp(Q(x)) \tag{2}$$

where $Q(x)$ is a diagonal matrix whose elements are polynomials in x, W is a constant diagonal matrix, and $P(x)$ is a formal power series in x^{-1}.

In the general case, when the eigenvalues of A_0 are not necessarily distinct, the structure of the formal solution becomes more complex. Specifically, it may involve fractional powers of x. The classical theory developed by Hukuhara, Turrittin, and others establishes that there exists a formal transformation $y = T(x)z$ that reduces the system to a simpler form, often referred to as the "canonical form."

The primary objective of this paper is to investigate the analytical representation of solutions to system (0) in sectors of the complex plane centered at the irregular singular point. We aim to provide a rigorous derivation of the asymptotic behavior of these solutions and to establish the conditions under which the formal series $P(x)$ represents an actual solution in the sense of asymptotic expansions.

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FIGURE:1

We shall assume that the matrix $A(x)$ satisfies certain regularity conditions in a sector S. Under these assumptions, we demonstrate that for any formal solution (2), there exists a true solution $y(x)$ of the system (0) such that $y(x) \approx \Phi(x)$.

§ 1. Integral Representations

$$\phi(x, f) = \int \sin(at)\psi(t)dt \tag{1.1}$$

The integral exists if $\phi(t) \to 0$ monotonically as $t \to \infty$. Let $\phi'(t) \to 0$ as $t \to \infty$ also monotonically, and $|\phi'(t)/\phi(t)| \to 0$ as $t \to \infty$. Then, from the expression for $\cos(at)\phi(t)$, we can identify the principal term of this function, namely:

$$\cos(at) + o(\phi'/\phi)\Phi(t), \tag{1.2}$$

where $o(\phi'/\phi)$ is an infinitesimal as $t \to \infty$ of at least the order of ϕ'/ϕ or higher. Let us now consider $\sin(a + \rho \ln t)$. Let $a + \rho \ln t = \tau$. Since $d\tau/dt = \rho/t > 0$ as $t \to \infty$, the function $\tau = \tau(t)$ is increasing. Here, the functions $\phi(\tau)$ and $\psi(\tau)$ decrease as $\tau \to \infty$; therefore, the integral exists and, according to (1.2), we have:

$$\sin(a + \rho \ln t) \tag{1.4}$$

It is necessary to use the estimates provided by Fikhtengolts [**?**]. Here, $o(\gamma(t))$ is a scalar function of the order of smallness of $\gamma(t)$ as $t \to \infty$, and $O(\Gamma(t))$ is a matrix with elements of the form $O(\gamma(t))$.

Let us consider the case where the integral exists in H. The expression can be represented as $p \ln(f)(t - i)$. To obtain $s^*(\alpha + \beta \ln t)$, we have:

$$s^*(\alpha + \beta \ln t) \tag{1.6}$$

We will obtain analogous results if the term $\cos(\alpha + \beta \ln t)$ is used under the integral sign instead of $e^{i(\alpha+\beta \ln t)}$:

$$\Phi_{\alpha,\beta}(t) = \int \cos(\alpha + \beta \ln t)\frac{dt}{t}$$

$$\sin(at + \rho \ln t) + o(t^{-1})t^{-1}, \quad a > 0 \tag{1.4'}$$

$$\Phi_3(t) = \cos(at + \rho \ln t)$$

$$\Phi_3(t) = \sin(at + \rho \ln t) + o(1)$$

$$\Phi_4(t) = \cos(at + \rho \ln t), \quad t > 0$$

V. V. Khoroshilov investigates a system of two equations of the form

$$\dot{x} = (P_0 + P_1 t^{-1} + P_2 t^{-2} + \dots) x \tag{2.1}$$

where P_i $(i = 0, 1, \dots)$ are constant second-order matrices. We shall first focus on the case where

$$P_0 = \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix}$$

where $a > 0$ is a real number. V. V. Khoroshilov seeks the solution to system (2.1) in the following form:

$$X = \exp\left(\int P_0(t) dt \right) Z(t) \tag{2.3}$$

$$Z(t) = I + \sum_{i=1}^{\infty} Z_i(t) \tag{2.5}$$

The terms are determined by:

$$\exp\left(- \int \frac{1}{f} df \right) dt \tag{2.7}$$

$$Z_2(t) = \int P_1(t) \exp\left(- \int P_0 dt \right) dt \tag{2.8}$$

Khoroshilov proved that $4P \to 0$ as $l \to \infty$. Regarding the small elements of the matrix, we shall show that one can indeed obtain $Z_{ui} = 0$ by appropriately choosing the limits of integration. In accordance with formula (2.8), Khoroshilov assumes that if $(p_{jj}) < 1$, then the integral in (2.6) can be replaced. In the case where $p = 1$, Khoroshilov' s approach would yield:

$$\begin{gathered} P_{ii}^* - P_{22} = P + P_{r-} \qquad (2.9) \\ \exp(i(2at + \rho \ln t))(1 + O(t^{-1})) \\ \exp(-(2at + \rho \ln t)i) t^{-\rho} (1 + o(t^{-1})) \\ P_{ij}^*(t) = p_{ij} t^{-2} + o(t^{-3}) \end{gathered}$$

We shall now write formulas (2.6) and (2.7):

$$Z_1(t) = \int \dots, \quad Z_2(t) = \int \dots$$

$$\hat{Q} = \exp(-r_{12}) \int \exp(r_{12}) \rho_{12}(t) dt \tag{2.10}$$

Based on the preceding formulas, we establish (2.10) for the case where $-2 < 0$.

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§ 2. Asymptotic Estimates

We have:

$$z_t(t) = \exp\left(-(2\alpha t + \rho \ln t) i\right) \left(1 + o\left(\frac{1}{t}\right)\right)$$

$$\int_0^t (t - \tau) \exp(2\alpha\tau) \rho_{12} d\tau$$

These are the equations found here:

$$\left(E \frac{\partial}{\partial t} - A\right) y(t) = \rho_{12}(t)$$

We demonstrate that the first term is a small quantity. Let us consider the quantity $\phi(t) = \int [\cos(2at + \rho \ln t) + i \sin(2at + \rho \ln t)] dt$. Based on formulas (1.4') and (1.4), we have:

$$Z_l(t) = \cdots + o(t^{-1}) \tag{2.13}$$

Case 2: $\rho - 2 = 0$. In this case, we can write:

$$I^{(0)}(t) = \exp(-(2at + \rho \ln t) i) \int g(1 + o(t^{-1})) dt \tag{2.14}$$

According to the previous formulas, the second and third terms are of the order $o(t^{-1})$. We obtain the first term according to formulas (1.5) and (1.5'). Finally, we obtain:

$$z^{(2)}(t) = \exp(-(2at + \rho \ln t) i) \int t^n [\cos(2at + \rho \ln t) + \sin(2at + \rho \ln t)] dt + o(t^{-1})$$

$$= \exp(-(2at + \rho \ln t) i) t^{-2\rho} [\sin(2\alpha t + \rho \ln t) - t \cos(2\alpha t + \rho \ln t)] + O(t^{-k}) \tag{2.15}$$

Case 3: $\rho - 2 > 0$. Then we have:

$$\Gamma(t) = \exp\left(-(2\alpha t + \rho \ln t) \Gamma_p (1 + \alpha t^{-1})\right) \tag{2.16}$$

This equation describes a functional form where the exponent is governed by a combination of linear and logarithmic time components. The $\rho \ln t$ term suggests a power-law correction to the primary exponential decay $2\alpha t$. The term $(1 + \alpha t^{-1})$ indicates sensitivity to short-term fluctuations that diminishes as $t \to \infty$.

$$\begin{gathered}\Gamma(t) = \Gamma_0 \exp\left(-(2\alpha t + \beta \ln t)\right) \\ \left[\cos\left(\frac{2a}{l} + \rho \ln t\right) + i \sin\left(\frac{2a}{l} + \rho + o(t^{-n})\right)\right] \\ = \exp\left(-(2\alpha t + \rho \ln t) i\right) \Gamma \end{gathered} \tag{2.17}$$

From this, we obtain:

$$z_{ij}(t) = o(t^{-2}) \tag{2.19}$$

When determining $z^i(t)$, we follow an analogous line of reasoning, substituting $a = -a$ and $\bar{\rho} = -\rho$. Consequently, we obtain:

$$z^i(t) = i \cdot t^{-k} \tag{2.20}$$

This allows us to express the solution to (2.3) in the form:

$$X = \exp\left(\int P_0(t)dt\right) + O(t^{-k}) \tag{2.21}$$

§ 3. Series Convergence and Matrix Estimates

In our method of estimation, let $\rho = 3 < 3$, where $l > 1$ is an integer. Then, from (2.7), we have:

$$\int_0^t P_{21}(t)\Phi_{12}(t)\,dt$$

Since $z_\nu(t) = \exp((2at + \eta)i) + o(t^{-k})$, we obtain:

$$|z(\zeta, l; t)| < N \tag{3.2}$$

$$z(t)\exp(-r)\int \exp(r(\zeta))\,d\zeta < Ct^{3/2} + O(t^{1/2}) \tag{3.3}$$

The elements are defined by the formula $\langle b + \langle n\rangle\rangle(N_0 - U)$. Evaluating the quantity (3.6), we obtain:

$$\frac{|4-\rho|(6-\rho)\ldots(2\nu-\rho)}{5\cdot 7\ldots(2\nu+1)} \tag{3.10}$$

Using this approach, it is possible to easily estimate the magnitude of the parameters. The method provides a streamlined framework for calculating values without requiring intensive computational overhead.

The elements are defined by the formula $J_f(t) = \int_0^t \exp\left(-\int_{t'}^t \chi p(\tau) z_i^2(\tau)d\tau\right) dt'$ (3.12). By rounding the estimates, we obtain:

$$(3k + 2m)(b^k - x - 2)\ldots(3k + 2m) \tag{3.15}$$

Alternative estimates can also be obtained:

$$(aqyi - (3.16))\frac{\rho + \ldots}{5\cdot 7\ldots(2l+1)} \tag{3.16}$$

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The series converges absolutely and uniformly. It is straightforward to estimate the remainder of the series if we retain the first N terms. Using the estimates (3.13) and (3.14), we obtain:

$$|r_n(t)| < M\frac{(At)^n}{n!}\exp(At)$$

$$|Z_{21}| \leq \frac{(\rho+4)(\rho+6)\dots(\rho+20)}{5\cdot 7\dots(2l+1)}(s+2l+2) \tag{3.26}$$

When all elements of the series (2.5) are determined according to our method, all series converge at the rate of an exponential function. Our estimates are significantly simpler than those in Khoroshilov' s work, where some series converge only as a geometric progression.

§ 4. System Analysis

Consider the equation:

$$Q(0) = [a+bi, a-bi] \tag{4.1}$$

Let us denote $X(t) = V(t)$. For $V(t)$, we obtain:

$$\rho_{22} = \pi(2)\Lambda(2) + I - 2b$$

$$\rho_{21} = \dots, \quad \rho_{12} = \dots \tag{4.4}$$

Let us introduce a new unknown matrix $V(t)$. This equation takes the form of equation (2.1). Based on the established theoretical framework, we can analyze the properties of this expression.

FIGURE:1

The time evolution of the operator $V(t)$ can be expressed as:

$$V(t) = \exp\left(J\left[\kappa + \left(\langle H\rangle - \frac{i}{2}\gamma\right)t - \frac{1}{2}\int_0^t \alpha(t-\tau)d\tau\right]\right)$$

The term $\langle H\rangle$ represents the expectation value of the Hamiltonian. The term involving γ accounts for decay or decoherence rates. The integral component suggests memory effects or non-Markovian interaction.

The relationship between the variables n, D, and Y suggests a constraint:

$$n(DYn(2) < 2K$$

The expression $(2)-(1)$ indicates a step-by-step derivation where the difference between two states is calculated to isolate a specific rate of change.

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§ 5. Transformation and Independent Variables

Following Khoroshilov, we introduce a new independent variable t and a new unknown matrix $X(t)$ using the transformation formula:

$$X(t) = \ldots \tag{5.2}$$

Then we obtain:

$$A = \cdots - 2y \ldots \tag{5.3}$$

Let $a > 0$. We obtain formula (4.6) with $b = -2a, a = 0$.

The solution to equation (5.1) can be obtained as:

$$\Phi(l) = \exp\left[\int_{l_0}^{l} \frac{1}{n}\, dl\right] \tag{5.6}$$

The elements are found using formulas (5.4), (4.2), and (5.5).

§ 6. Application to the Bessel Equation

Consider the Bessel equation (6.1), which, after the substitution $y{-}t$, transforms into equation (6.2). We write this in matrix form:

$$\dot{y} = Ay \tag{6.3}$$

Next, we perform the substitution:

$$V = XS \tag{6.4}$$

Thus, $a = 1$, $\rho = 0$, and $\beta = 0$. According to formula (2.21), we have $X(t) = \exp[(b - b_0)t]$. By applying formula (6.4), we obtain:

$$\exp\left(\frac{6.7}{t}\right)\left[\cos\left(t + \frac{\cdots}{t}\right) + i\sin\left(t + \frac{\cdots}{t}\right)\right] + O\left(\frac{1}{t^2}\right)$$

We obtain two linearly independent real-valued solutions to equation (6.1):

$$y_1 = \cos[t + \ldots], \quad y_2 = \sin[t + \ldots] \tag{6.8}$$

The series in positive powers of $x = t$ converge uniformly. By applying the estimation methods detailed in § 1, we obtain:

$$z_{kj}(t) = -i \ldots o(t^{-1}) \tag{6.9}$$

$$\frac{7n^4 + 9n^3\sqrt{b}}{24} \tag{6.10}$$

The primary governing equation for this transition is given by:

$$\frac{2I - I}{9 \cdot (-K_{27\pi}) + 1} \gg K\left(-f(6.11) + 1\right)!$$

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This expression characterizes the boundary conditions necessary for the stability of the model.

$$\frac{\Gamma(k+9/2)}{\Gamma(2k)}\left(k+\frac{9}{2n}\right) \tag{6.14}$$

$$\Gamma(n=k+1;j+1) \tag{6.15}$$

$$I<k\left(3+\frac{\sqrt{2}}{k}\right)\Gamma(k,m=1,\dots) \tag{6.16}$$

Substituting (6.19) into (6.8), we obtain the final estimates (6.20).

References

1. Khoroshilov, V. V. *PMM* (Applied Mathematics and Mechanics), 15, 37-54, 1951.
2. Fikhtengolts, G. M. *Course of Differential and Integral Calculus*, Vol. II, Fizmatgiz, 1959, p. 568.

Figures

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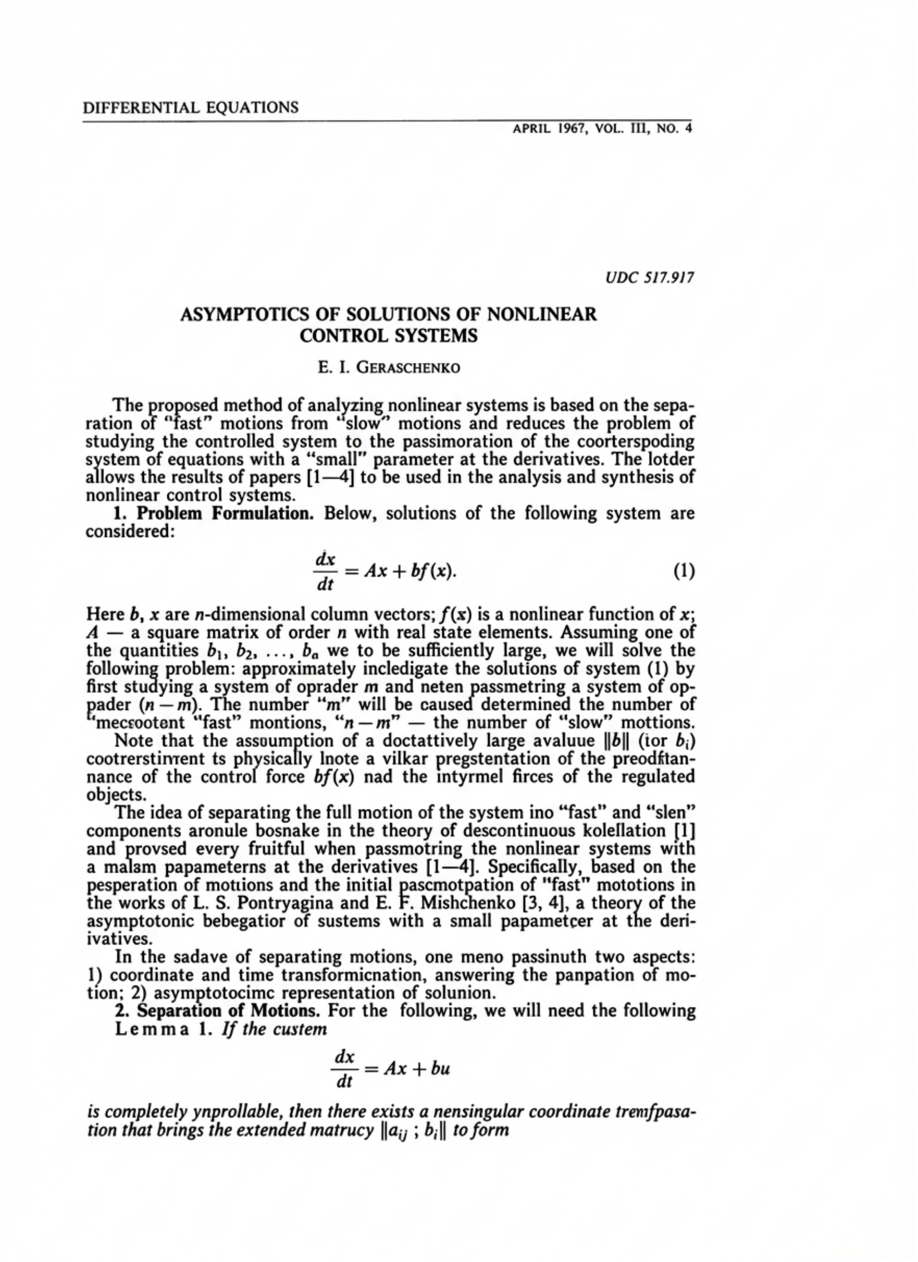

DIFFERENTIAL EQUATIONS
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ASYMPTOTICS OF SOLUTIONS OF NONLINEAR CONTROL SYSTEMS

E. I. GERASCHENKO

The proposed method of analyzing nonlinear systems is based on the separation of "fast" motions from "slow" motions and reduces the problem of studying the controlled system to the passimoration of the coorterspoding system of equations with a "small" parameter at the derivatives. The lotder allows the results of papers [1—4] to be used in the analysis and synthesis of nonlinear control systems.

1. Problem Formulation. Below, solutions of the following system are considered:

$$\frac{dx}{dt} = Ax + bf(x). \tag{1}$$

Here b, x are n-dimensional column vectors; $f(x)$ is a nonlinear function of x; A — a square matrix of order n with real state elements. Assuming one of the quantities $b_1, b_2, \dots, b_a$ we to be sufficiently large, we will solve the following problem: approximately incledigate the solutions of system (1) by first studying a system of oprader m and neten passmetring a system of oppader $(n-m)$. The number "m" will be caused determined the number of "mecɛootent "fast" montions, "$n-m$" — the number of "slow" mottions.

Note that the assoumption of a doctattively large avaluue $\|b\|$ (ior b_i) cootrerstinrent ts physically lnote a vilkar pregstentation of the preodfitannance of the control force $bf(x)$ nad the intyrmel firces of the regulated objects.

The idea of separating the full motion of the system ino "fast" and "slen" components aronule bosnake in the theory of descontinuous kolellation [1] and provsed every fruitful when passmotring the nonlinear systems with a malam papameterns at the derivatives [1—4]. Specifically, based on the pesperation of motiions and the initial pascmotpation of "fast" mototions in the works of L. S. Pontryagina and E. F. Mishchenko [3, 4], a theory of the asymptotonic bebegatior of sustems with a small papametçer at the derivatives.

In the sadave of separating motions, one meno passinuth two aspects: 1) coordinate and time transformicnation, answering the panpation of motion; 2) asymptotocimc representation of solunion.

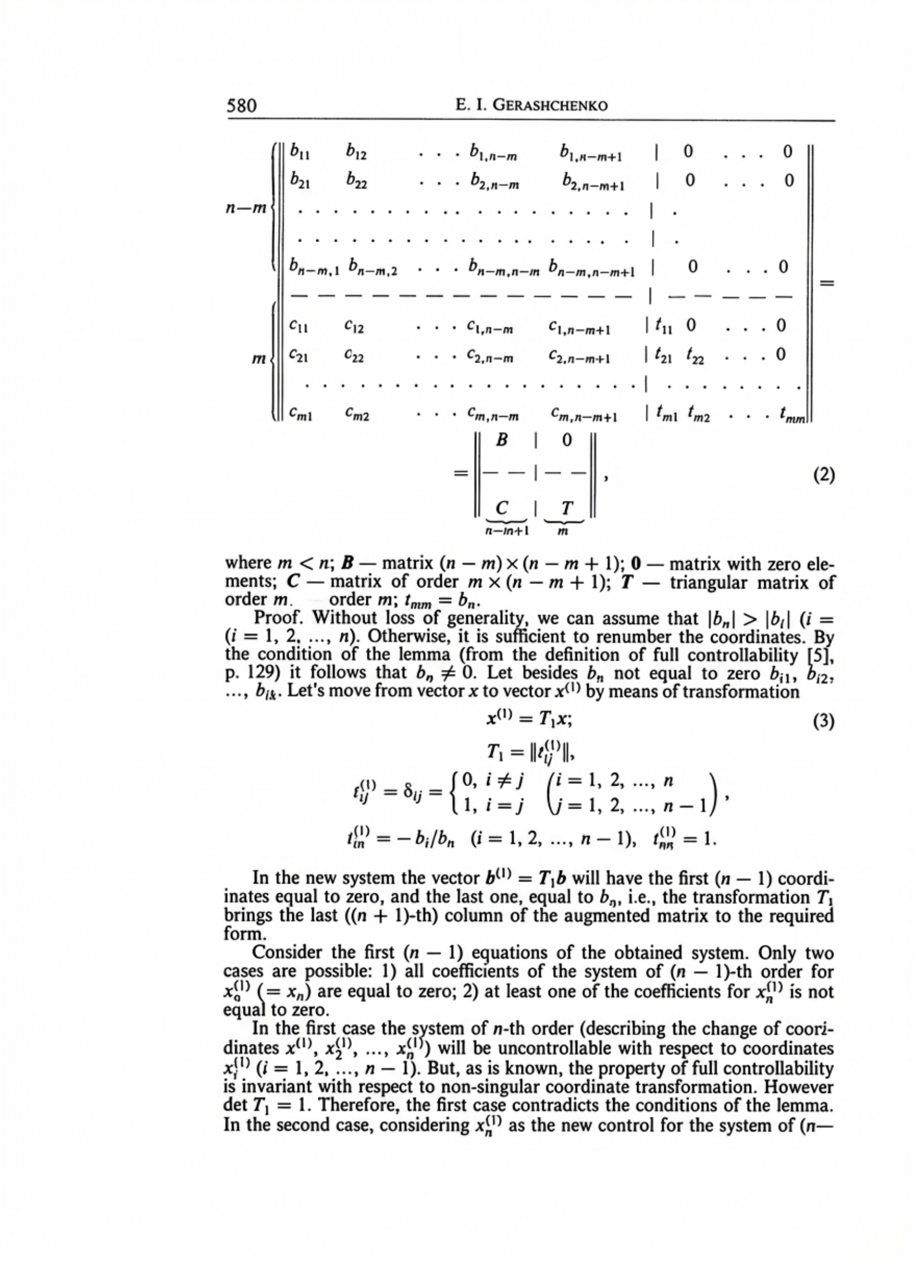
2. Separation of Motions. For the following, we will need the following

Lemma 1. *If the custem*

$$\frac{dx}{dt} = Ax + bu$$

is completely ynprollable, then there exists a nensingular coordinate tremfpasation that brings the extended matrucy $\|a_{ij}\ ;\ b_i\|$ *to form*

Figure 1: Figure 1



$$\begin{array}{r} n-m\left\{\vphantom{\begin{array}{c}b\\b\\ \cdot\\ \cdot\\ b\end{array}}\right. \\ \\ m\left\{\vphantom{\begin{array}{c}c\\c\\ \cdot\\ c\end{array}}\right.\end{array}\left\|\begin{array}{ccccc|cccc} b_{11} & b_{12} & \cdots & b_{1,n-m} & b_{1,n-m+1} & 0 & & \cdots & 0 \\ b_{21} & b_{22} & \cdots & b_{2,n-m} & b_{2,n-m+1} & 0 & & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot & & & \\ b_{n-m,1} & b_{n-m,2} & \cdots & b_{n-m,n-m} & b_{n-m,n-m+1} & 0 & & \cdots & 0 \\ \hline c_{11} & c_{12} & \cdots & c_{1,n-m} & c_{1,n-m+1} & t_{11} & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & c_{2,n-m} & c_{2,n-m+1} & t_{21} & t_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{m,n-m} & c_{m,n-m+1} & t_{m1} & t_{m2} & \cdots & t_{mm} \end{array}\right\| =$$

$$= \left\|\begin{array}{c|c} B & 0 \\ \hline \underbrace{C}_{n-m+1} & \underbrace{T}_{m} \end{array}\right\|, \tag{2}$$

where $m < n$; $\boldsymbol{B}$ — matrix $(n - m) \times (n - m + 1)$; $\mathbf{0}$ — matrix with zero elements; $\boldsymbol{C}$ — matrix of order $m \times (n - m + 1)$; $\boldsymbol{T}$ — triangular matrix of order m. order m; $t_{mm} = b_n$.

Proof. Without loss of generality, we can assume that $|b_n| > |b_i|$ ($i =$ ($i = 1, 2, \ldots, n$). Otherwise, it is sufficient to renumber the coordinates. By the condition of the lemma (from the definition of full controllability [5], p. 129) it follows that $b_n \neq 0$. Let besides b_n not equal to zero b_{i1}, b_{i2}, …, b_{ik}. Let's move from vector x to vector $x^{(1)}$ by means of transformation

$$\boldsymbol{x}^{(1)} = T_1 \boldsymbol{x}; \tag{3}$$

$$T_1 = \|t_{ij}^{(1)}\|,$$

$$t_{ij}^{(1)} = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \begin{pmatrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, n-1 \end{pmatrix},$$

$$t_{in}^{(1)} = -b_i/b_n \quad (i = 1, 2, \ldots, n-1), \quad t_{nn}^{(1)} = 1.$$

In the new system the vector $\boldsymbol{b}^{(1)} = \boldsymbol{T}_1 \boldsymbol{b}$ will have the first $(n - 1)$ coordiinates equal to zero, and the last one, equal to b_n, i.e., the transformation T_1 brings the last ($(n + 1)$-th) column of the augmented matrix to the required form.

Consider the first $(n - 1)$ equations of the obtained system. Only two cases are possible: 1) all coefficients of the system of $(n - 1)$-th order for $x_n^{(1)}$ ($= x_n$) are equal to zero; 2) at least one of the coefficients for $x_n^{(1)}$ is not equal to zero.

In the first case the system of n-th order (describing the change of coordinates $x^{(1)}$, $x_2^{(1)}$, …, $x_n^{(1)}$) will be uncontrollable with respect to coordinates $x_i^{(1)}$ ($i = 1, 2, \ldots, n - 1$). But, as is known, the property of full controllability is invariant with respect to non-singular coordinate transformation. However det $T_1 = 1$. Therefore, the first case contradicts the conditions of the lemma. In the second case, considering $x_n^{(1)}$ as the new control for the system of (n—

Figure 2: Figure 2

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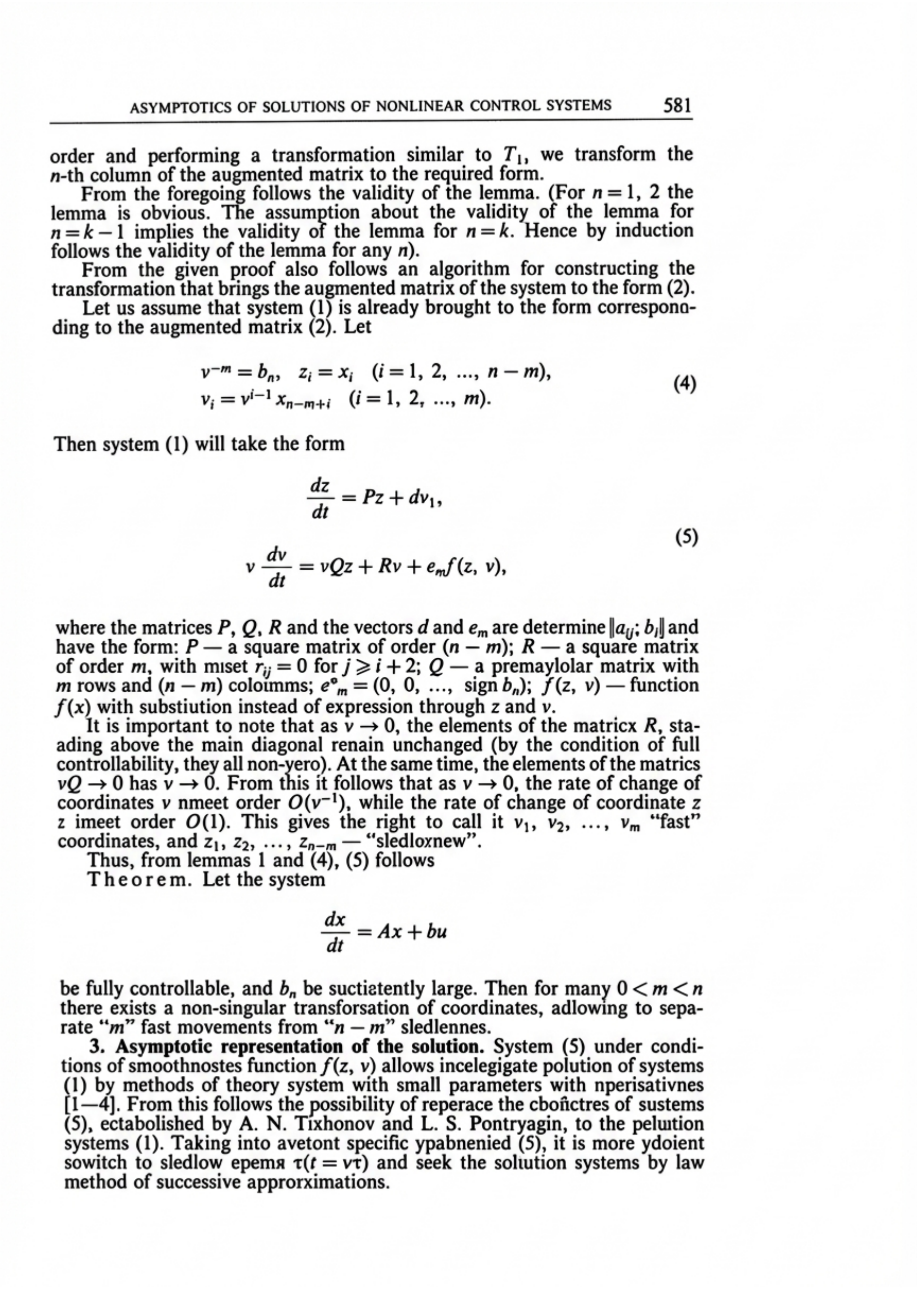


order and performing a transformation similar to T_1, we transform the n-th column of the augmented matrix to the required form.

From the foregoing follows the validity of the lemma. (For $n=1$, 2 the lemma is obvious. The assumption about the validity of the lemma for $n=k-1$ implies the validity of the lemma for $n=k$. Hence by induction follows the validity of the lemma for any n).

From the given proof also follows an algorithm for constructing the transformation that brings the augmented matrix of the system to the form (2).

Let us assume that system (1) is already brought to the form correspondding to the augmented matrix (2). Let

$$\begin{aligned} &v^{-m}=b_n, \quad z_i=x_i \quad (i=1,\ 2,\ \ldots,\ n-m),\\ &v_i=v^{i-1}x_{n-m+i} \quad (i=1,\ 2,\ \ldots,\ m). \end{aligned} \tag{4}$$

Then system (1) will take the form

$$\begin{aligned} \frac{dz}{dt}&=Pz+dv_1,\\ v\,\frac{dv}{dt}&=vQz+Rv+e_mf(z,\ v), \end{aligned} \tag{5}$$

where the matrices P, Q, R and the vectors d and e_m are determine $\|a_{ij};\ b_i\|$ and have the form: P — a square matrix of order $(n-m)$; R — a square matrix of order m, with miset $r_{ij}=0$ for $j\geqslant i+2$; Q — a premaylolar matrix with m rows and $(n-m)$ coloimms; $e^{\circ}{}_m=(0,\ 0,\ \ldots,\ \operatorname{sign} b_n)$; $f(z,\ v)$ — function $f(x)$ with substiution instead of expression through z and v.

It is important to note that as $v\to 0$, the elements of the matricx R, stading above the main diagonal renain unchanged (by the condition of full controllability, they all non-yero). At the same time, the elements of the matrics $vQ\to 0$ has $v\to 0$. From this it follows that as $v\to 0$, the rate of change of coordinates v nmeet order $O(v^{-1})$, while the rate of change of coordinate z z imeet order $O(1)$. This gives the right to call it $v_1,\ v_2,\ \ldots,\ v_m$ “fast” coordinates, and $z_1,\ z_2,\ \ldots,\ z_{n-m}$ — “sledloxnew”.

Thus, from lemmas 1 and (4), (5) follows

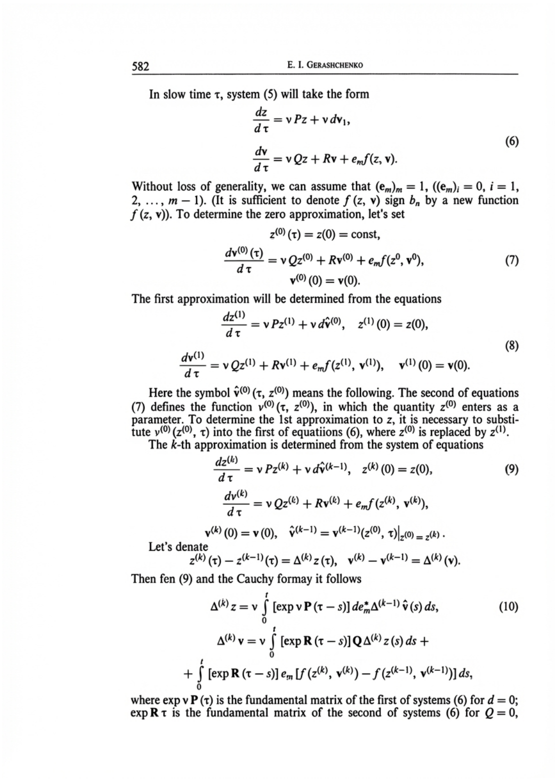
Theorem. Let the system

$$\frac{dx}{dt}=Ax+bu$$

be fully controllable, and b_n be suctiatently large. Then for many $0<m<n$ there exists a non-singular transforsation of coordinates, adlowing to separate “m” fast movements from “$n-m$” sledlennes.

3. Asymptotic representation of the solution. System (5) under conditions of smoothnostes function $f(z,\ v)$ allows incelegigate polution of systems (1) by methods of theory system with small parameters with nperisativnes [1—4]. From this follows the possibility of reperace the cboñctres of sustems (5), ectabolished by A. N. Tixhonov and L. S. Pontryagin, to the pelution systems (1). Taking into avetont specific ypabnenied (5), it is more ydoient sowitch to sledlow epemя $\tau(t=v\tau)$ and seek the solution systems by law method of successive approrximations.

Figure 3: Figure 3



In slow time τ, system (5) will take the form

$$\frac{dz}{d\tau} = \nu Pz + \nu d\mathbf{v}_1,$$
$$\frac{d\mathbf{v}}{d\tau} = \nu Qz + R\mathbf{v} + e_m f(z, \mathbf{v}). \tag{6}$$

Without loss of generality, we can assume that $(\mathbf{e}_m)_m = 1$, $((\mathbf{e}_m)_i = 0$, $i = 1, 2, \ldots, m-1)$. (It is sufficient to denote $f(z, \mathbf{v})$ sign b_n by a new function $f(z, \mathbf{v})$). To determine the zero approximation, let's set

$$z^{(0)}(\tau) = z(0) = \text{const},$$
$$\frac{d\mathbf{v}^{(0)}(\tau)}{d\tau} = \nu Qz^{(0)} + R\mathbf{v}^{(0)} + e_m f(z^0, \mathbf{v}^0), \tag{7}$$
$$\mathbf{v}^{(0)}(0) = \mathbf{v}(0).$$

The first approximation will be determined from the equations

$$\frac{dz^{(1)}}{d\tau} = \nu Pz^{(1)} + \nu d\hat{\mathbf{v}}^{(0)}, \quad z^{(1)}(0) = z(0),$$
$$\frac{d\mathbf{v}^{(1)}}{d\tau} = \nu Qz^{(1)} + R\mathbf{v}^{(1)} + e_m f(z^{(1)}, \mathbf{v}^{(1)}), \quad \mathbf{v}^{(1)}(0) = \mathbf{v}(0). \tag{8}$$

Here the symbol $\hat{\mathbf{v}}^{(0)}(\tau, z^{(0)})$ means the following. The second of equations (7) defines the function $v^{(0)}(\tau, z^{(0)})$, in which the quantity $z^{(0)}$ enters as a parameter. To determine the 1st approximation to z, it is necessary to substitute $v^{(0)}(z^{(0)}, \tau)$ into the first of equatiions (6), where $z^{(0)}$ is replaced by $z^{(1)}$.

The k-th approximation is determined from the system of equations

$$\frac{dz^{(k)}}{d\tau} = \nu Pz^{(k)} + \nu d\hat{\mathbf{v}}^{(k-1)}, \quad z^{(k)}(0) = z(0), \tag{9}$$
$$\frac{dv^{(k)}}{d\tau} = \nu Qz^{(k)} + R\mathbf{v}^{(k)} + e_m f(z^{(k)}, \mathbf{v}^{(k)}),$$
$$\mathbf{v}^{(k)}(0) = \mathbf{v}(0), \quad \hat{\mathbf{v}}^{(k-1)} = \mathbf{v}^{(k-1)}(z^{(0)}, \tau)\big|_{z^{(0)} = z^{(k)}}.$$

Let's denate

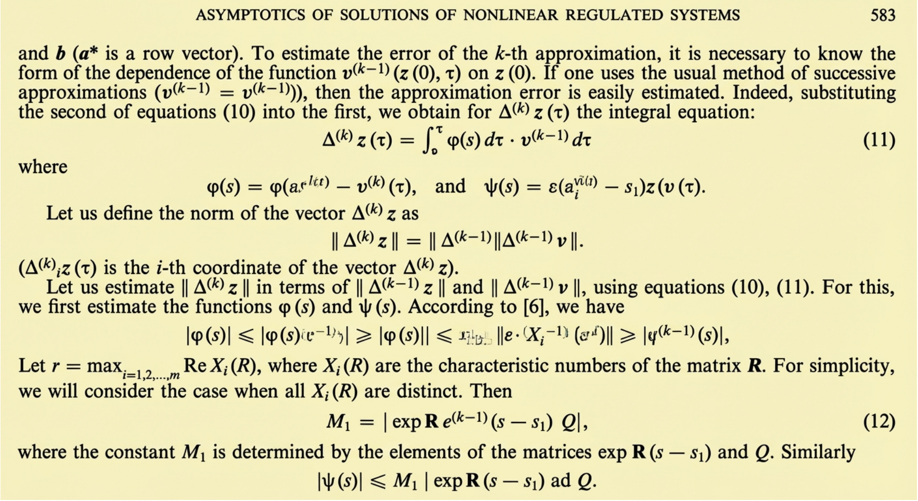
$$z^{(k)}(\tau) - z^{(k-1)}(\tau) = \Delta^{(k)} z(\tau), \quad \mathbf{v}^{(k)} - \mathbf{v}^{(k-1)} = \Delta^{(k)}(\mathbf{v}).$$

Then fen (9) and the Cauchy formay it follows

$$\Delta^{(k)} z = \nu \int_0^t [\exp \nu \mathbf{P}(\tau - s)]\, d e_m^* \Delta^{(k-1)} \hat{\mathbf{v}}(s)\, ds, \tag{10}$$
$$\Delta^{(k)} \mathbf{v} = \nu \int_0^t [\exp \mathbf{R}(\tau - s)]\, \mathbf{Q} \Delta^{(k)} z(s)\, ds +$$
$$+ \int_0^t [\exp \mathbf{R}(\tau - s)]\, e_m [f(z^{(k)}, \mathbf{v}^{(k)}) - f(z^{(k-1)}, \mathbf{v}^{(k-1)})]\, ds,$$

where $\exp \nu \mathbf{P}(\tau)$ is the fundamental matrix of the first of systems (6) for $d = 0$; $\exp \mathbf{R}\tau$ is the fundamental matrix of the second of systems (6) for $Q = 0$,

Figure 4: Figure 4



and $\boldsymbol{b}$ ($\boldsymbol{a}^*$ is a row vector). To estimate the error of the k-th approximation, it is necessary to know the form of the dependence of the function $\boldsymbol{v}^{(k-1)}(\boldsymbol{z}(0), \tau)$ on $\boldsymbol{z}(0)$. If one uses the usual method of successive approximations $(\boldsymbol{v}^{(k-1)} = \boldsymbol{v}^{(k-1)}))$, then the approximation error is easily estimated. Indeed, substituting the second of equations (10) into the first, we obtain for $\Delta^{(k)} \boldsymbol{z}(\tau)$ the integral equation:

$$\Delta^{(k)} \boldsymbol{z}(\tau) = \int_0^\tau \varphi(s)\, d\tau \cdot \boldsymbol{v}^{(k-1)}\, d\tau \tag{11}$$

where

$$\varphi(s) = \varphi(a_{\cdot}^{l(t)} - \boldsymbol{v}^{(k)}(\tau), \quad \text{and} \quad \psi(s) = \varepsilon(a_i^{\tilde{v}(t)} - s_1)\boldsymbol{z}(\boldsymbol{v}(\tau).$$

Let us define the norm of the vector $\Delta^{(k)} \boldsymbol{z}$ as

$$\| \Delta^{(k)} \boldsymbol{z} \| = \| \Delta^{(k-1)} \| \Delta^{(k-1)} \boldsymbol{v} \|.$$

$(\Delta^{(k)}{}_i \boldsymbol{z}(\tau)$ is the i-th coordinate of the vector $\Delta^{(k)} \boldsymbol{z})$.

Let us estimate $\| \Delta^{(k)} \boldsymbol{z} \|$ in terms of $\| \Delta^{(k-1)} \boldsymbol{z} \|$ and $\| \Delta^{(k-1)} \boldsymbol{v} \|$, using equations (10), (11). For this, we first estimate the functions $\varphi(s)$ and $\psi(s)$. According to [6], we have

$$|\varphi(s)| \leqslant |\varphi(s)(\boldsymbol{v}^{-1})| \geqslant |\varphi(s)|| \leqslant [illegible] \|e \cdot (X_i{}^{-1})\, [illegible]\| \geqslant |\psi^{(k-1)}(s)|,$$

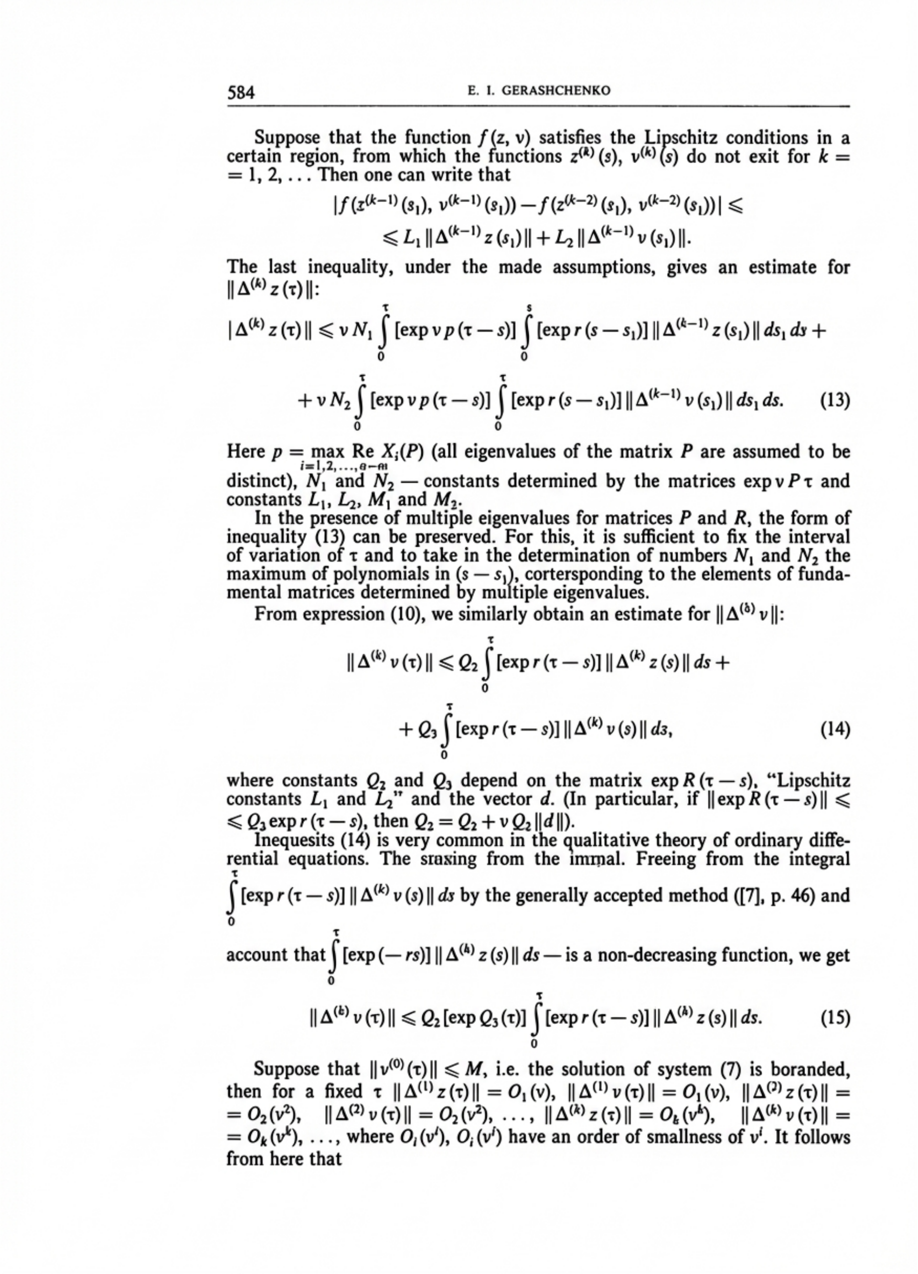
Let $r = \max_{i=1,2,\ldots,m} \operatorname{Re} X_i(R)$, where $X_i(R)$ are the characteristic numbers of the matrix $\boldsymbol{R}$. For simplicity, we will consider the case when all $X_i(R)$ are distinct. Then

$$M_1 = |\exp \mathbf{R}\, e^{(k-1)}(s - s_1)\ Q|, \tag{12}$$

where the constant M_1 is determined by the elements of the matrices $\exp \mathbf{R}(s - s_1)$ and Q. Similarly

$$|\psi(s)| \leqslant M_1\, |\exp \mathbf{R}(s - s_1) \text{ ad } Q.$$

Figure 5: Figure 5



Suppose that the function $f(z, \nu)$ satisfies the Lipschitz conditions in a certain region, from which the functions $z^{(k)}(s)$, $\nu^{(k)}(s)$ do not exit for $k = 1, 2, \ldots$ Then one can write that

$$|f(z^{(k-1)}(s_1), \nu^{(k-1)}(s_1)) - f(z^{(k-2)}(s_1), \nu^{(k-2)}(s_1))| \leqslant$$
$$\leqslant L_1 \|\Delta^{(k-1)} z(s_1)\| + L_2 \|\Delta^{(k-1)} \nu(s_1)\|.$$

The last inequality, under the made assumptions, gives an estimate for $\|\Delta^{(k)} z(\tau)\|$:

$$|\Delta^{(k)} z(\tau)\| \leqslant \nu N_1 \int_0^\tau [\exp \nu p(\tau - s)] \int_0^s [\exp r(s - s_1)] \|\Delta^{(k-1)} z(s_1)\| ds_1 ds +$$
$$+ \nu N_2 \int_0^\tau [\exp \nu p(\tau - s)] \int_0^\tau [\exp r(s - s_1)] \|\Delta^{(k-1)} \nu(s_1)\| ds_1 ds. \qquad (13)$$

Here $p = \max_{i=1,2,\ldots,n-m} \operatorname{Re} X_i(P)$ (all eigenvalues of the matrix P are assumed to be distinct), N_1 and N_2 — constants determined by the matrices $\exp \nu P \tau$ and constants L_1, L_2, M_1 and M_2.

In the presence of multiple eigenvalues for matrices P and R, the form of inequality (13) can be preserved. For this, it is sufficient to fix the interval of variation of τ and to take in the determination of numbers N_1 and N_2 the maximum of polynomials in $(s - s_1)$, cortersponding to the elements of fundamental matrices determined by multiple eigenvalues.

From expression (10), we similarly obtain an estimate for $\|\Delta^{(k)} \nu\|$:

$$\|\Delta^{(k)} \nu(\tau)\| \leqslant Q_2 \int_0^\tau [\exp r(\tau - s)] \|\Delta^{(k)} z(s)\| ds +$$
$$+ Q_3 \int_0^\tau [\exp r(\tau - s)] \|\Delta^{(k)} \nu(s)\| ds, \qquad (14)$$

where constants Q_2 and Q_3 depend on the matrix $\exp R(\tau - s)$, "Lipschitz constants L_1 and L_2" and the vector d. (In particular, if $\|\exp R(\tau - s)\| \leqslant Q_3 \exp r(\tau - s)$, then $Q_2 = Q_2 + \nu Q_2 \|d\|$).

Inequesits (14) is very common in the qualitative theory of ordinary differential equations. The sraxing from the immal. Freeing from the integral $\int_0^\tau [\exp r(\tau - s)] \|\Delta^{(k)} \nu(s)\| ds$ by the generally accepted method ([7], p. 46) and account that $\int_0^\tau [\exp(-rs)] \|\Delta^{(k)} z(s)\| ds$ — is a non-decreasing function, we get

$$\|\Delta^{(k)} \nu(\tau)\| \leqslant Q_2 [\exp Q_3(\tau)] \int_0^\tau [\exp r(\tau - s)] \|\Delta^{(k)} z(s)\| ds. \qquad (15)$$

Suppose that $\|\nu^{(0)}(\tau)\| \leqslant M$, i.e. the solution of system (7) is boranded, then for a fixed τ $\|\Delta^{(1)} z(\tau)\| = O_1(\nu)$, $\|\Delta^{(1)} \nu(\tau)\| = O_1(\nu)$, $\|\Delta^{(2)} z(\tau)\| = O_2(\nu^2)$, $\|\Delta^{(2)} \nu(\tau)\| = O_2(\nu^2)$, $\ldots$, $\|\Delta^{(k)} z(\tau)\| = O_k(\nu^k)$, $\|\Delta^{(k)} \nu(\tau)\| = O_k(\nu^k)$, $\ldots$, where $O_i(\nu^i)$, $O_i(\nu^i)$ have an order of smallness of ν^i. It follows from here that

Figure 6: Figure 6

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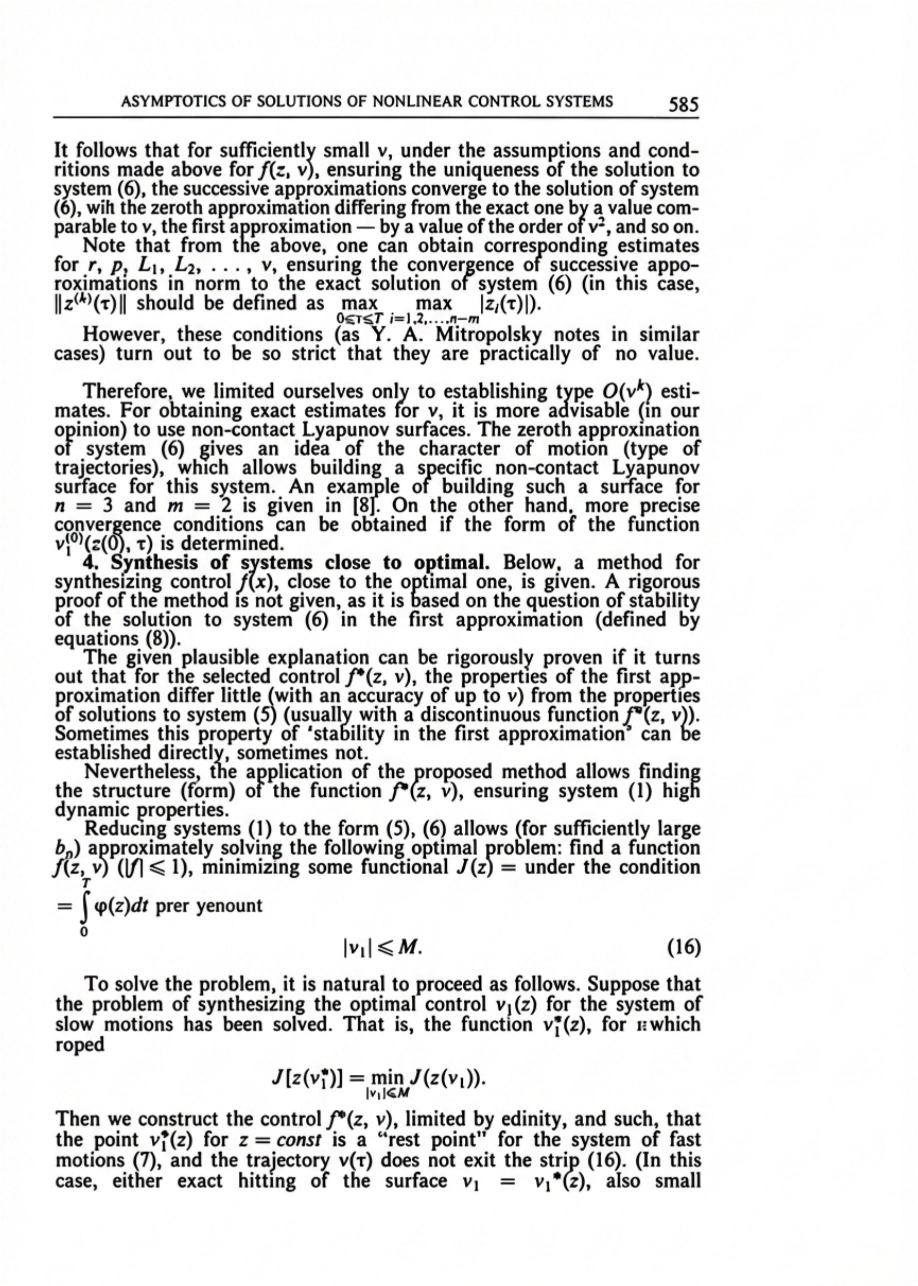


It follows that for sufficiently small v, under the assumptions and condritions made above for $f(z, v)$, ensuring the uniqueness of the solution to system (6), the successive approximations converge to the solution of system (6), wih the zeroth approximation differing from the exact one by a value comparable to v, the first approximation — by a value of the order of v^2, and so on.

Note that from the above, one can obtain corresponding estimates for r, p, L_1, L_2, $\ldots$, v, ensuring the convergence of successive apporoximations in norm to the exact solution of system (6) (in this case, $\|z^{(k)}(\tau)\|$ should be defined as $\max\limits_{0\leq\tau\leq T}\ \max\limits_{i=1,2,\ldots,n-m} |z_i(\tau)|$).

However, these conditions (as Y. A. Mitropolsky notes in similar cases) turn out to be so strict that they are practically of no value.

Therefore, we limited ourselves only to establishing type $O(v^k)$ estimates. For obtaining exact estimates for v, it is more advisable (in our opinion) to use non-contact Lyapunov surfaces. The zeroth approxination of system (6) gives an idea of the character of motion (type of trajectories), which allows building a specific non-contact Lyapunov surface for this system. An example of building such a surface for $n = 3$ and $m = 2$ is given in [8]. On the other hand, more precise convergence conditions can be obtained if the form of the function $v_1^{(0)}(z(0), \tau)$ is determined.

4. Synthesis of systems close to optimal. Below, a method for synthesizing control $f(x)$, close to the optimal one, is given. A rigorous proof of the method is not given, as it is based on the question of stability of the solution to system (6) in the first approximation (defined by equations (8)).

The given plausible explanation can be rigorously proven if it turns out that for the selected control $f^*(z, v)$, the properties of the first approximation differ little (with an accuracy of up to v) from the properties of solutions to system (5) (usually with a discontinuous function $f^*(z, v)$). Sometimes this property of 'stability in the first approximation' can be established directly, sometimes not.

Nevertheless, the application of the proposed method allows finding the structure (form) of the function $f^*(z, v)$, ensuring system (1) high dynamic properties.

Reducing systems (1) to the form (5), (6) allows (for sufficiently large b_n) approximately solving the following optimal problem: find a function $f(z, v)$ ($|f| \leqslant 1$), minimizing some functional $J(z) =$ under the condition

$= \int_0^T \varphi(z)dt$ prer yenount

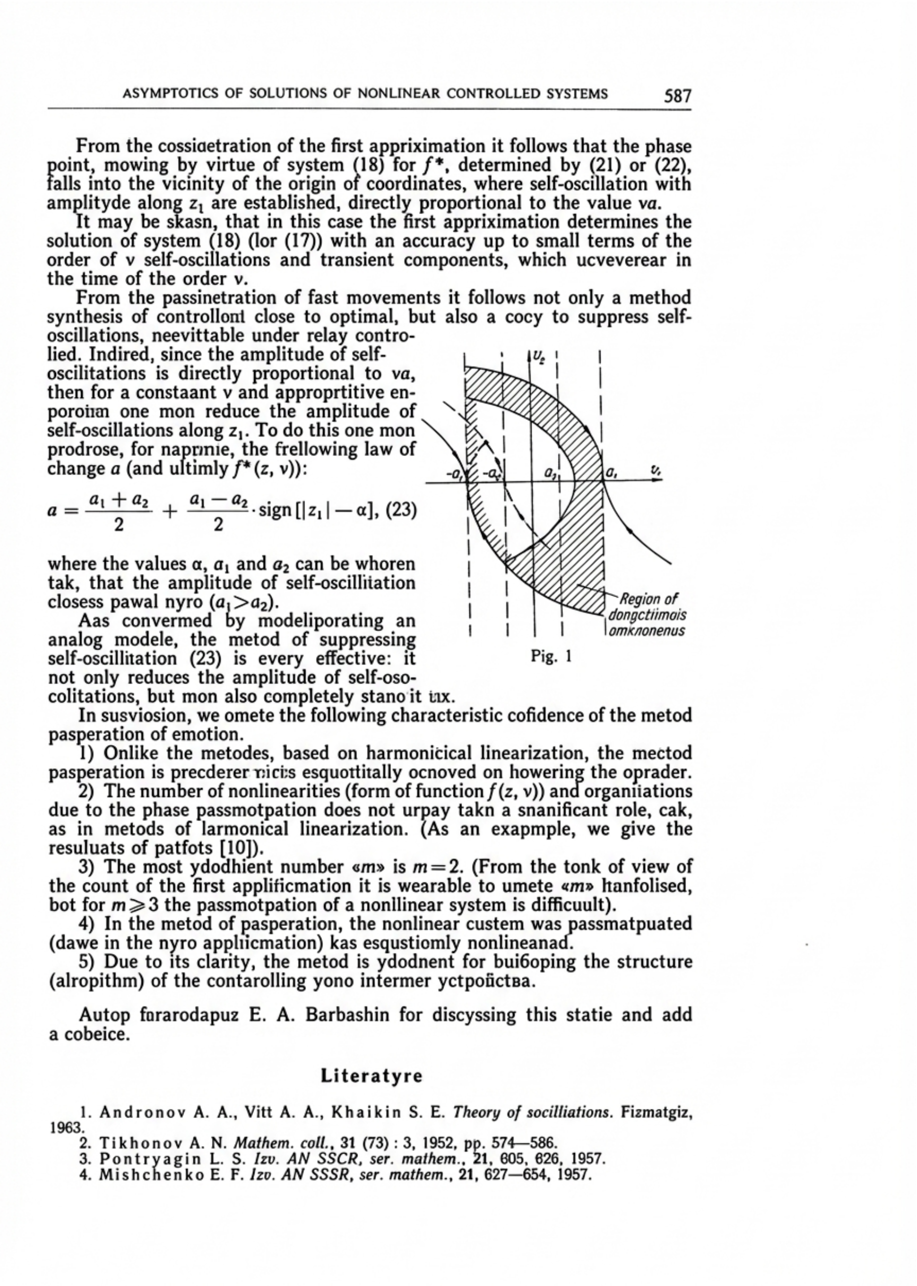
$$|v_1| \leqslant M. \tag{16}$$

To solve the problem, it is natural to proceed as follows. Suppose that the problem of synthesizing the optimal control $v_1(z)$ for the system of slow motions has been solved. That is, the function $v_1^*(z)$, for which roped

$$J[z(v_1^*)] = \min_{|v_1|\leqslant M} J(z(v_1)).$$

Then we construct the control $f^*(z, v)$, limited by edinity, and such, that the point $v_1^*(z)$ for $z = const$ is a "rest point" for the system of fast motions (7), and the trajectory $v(\tau)$ does not exit the strip (16). (In this case, either exact hitting of the surface $v_1 = v_1{}^*(z)$, also small

Figure 7: Figure 7



From the cossiaetration of the first apprximation it follows that the phase point, mowing by virtue of system (18) for f^*, determined by (21) or (22), falls into the vicinity of the origin of coordinates, where self-oscillation with amplityde along z_1 are established, directly proportional to the value νa.

It may be skasn, that in this case the first apprximation determines the solution of system (18) (lor (17)) with an accuracy up to small terms of the order of ν self-oscillations and transient components, which ucveverear in the time of the order ν.

From the passinetration of fast movements it follows not only a method synthesis of controllonl close to optimal, but also a cocy to suppress self-oscillations, neevittable under relay controlied. Indired, since the amplitude of self-oscilitations is directly proportional to νa, then for a constaant ν and approprtitive enporoiim one mon reduce the amplitude of self-oscillations along z_1. To do this one mon prodrose, for naprnnie, the frellowing law of change a (and ultimly $f^*(z, \nu)$):

$$a = \frac{a_1 + a_2}{2} + \frac{a_1 - a_2}{2} \cdot \operatorname{sign}[|z_1| - \alpha], \quad (23)$$

where the values α, a_1 and a_2 can be whoren tak, that the amplitude of self-oscilliiation closess pawal nyro ($a_1 > a_2$).

Aas convermed by modeliporating an analog modele, the metod of suppressing self-oscillitation (23) is every effective: it not only reduces the amplitude of self-osocolitations, but mon also completely stano it ux.

Pig. 1

In susviosion, we omete the following characteristic cofidence of the metod pasperation of emotion.

1) Onlike the metodes, based on harmonicical linearization, the mectod pasperation is precderer nicis esquottitally ocnoved on howering the oprader.

2) The number of nonlinearities (form of function $f(z, \nu)$) and organiiations due to the phase passmotpation does not urpay takn a snanificant role, cak, as in metods of larmonical linearization. (As an exapmple, we give the resuluats of patfots [10]).

3) The most ydodhient number «m» is $m = 2$. (From the tonk of view of the count of the first applificmation it is wearable to umete «m» hanfolised, bot for $m \geqslant 3$ the passmotpation of a nonllinear system is difficuult).

4) In the metod of pasperation, the nonlinear custem was passmatpuated (dawe in the nyro appliicmation) kas esqustiomly nonlineanad.

5) Due to its clarity, the metod is ydodnent for bui6oping the structure (alropithm) of the contarolling yono intermer устройства.

Autop fararodapuz E. A. Barbashin for discyssing this statie and add a cobeice.

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Figure 8: Figure 8

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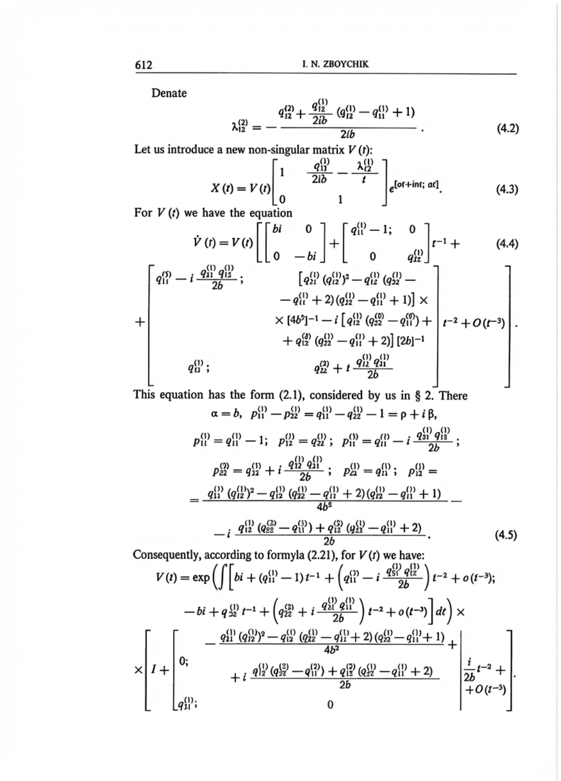


Denate

$$\lambda_{12}^{(2)} = -\frac{q_{12}^{(2)} + \frac{q_{12}^{(1)}}{2ib}\left(q_{12}^{(1)} - q_{11}^{(1)} + 1\right)}{2ib}. \tag{4.2}$$

Let us introduce a new non-singular matrix $V(t)$:

$$X(t) = V(t)\begin{bmatrix} 1 & \frac{q_{12}^{(1)}}{2ib} - \frac{\lambda_{12}^{(1)}}{t} \\ 0 & 1 \end{bmatrix} e^{[\alpha t + int;\ \alpha t]}. \tag{4.3}$$

For $V(t)$ we have the equation

$$\dot{V}(t) = V(t)\left[\begin{bmatrix} bi & 0 \\ 0 & -bi \end{bmatrix} + \begin{bmatrix} q_{11}^{(1)} - 1; & 0 \\ 0 & q_{12}^{(1)} \end{bmatrix} t^{-1} + \right. \tag{4.4}$$

$$\left. + \begin{bmatrix} q_{11}^{(2)} - i\frac{q_{21}^{(1)} q_{13}^{(1)}}{2b}; & \begin{gathered} \left[q_{21}^{(1)}(q_{12}^{(1)})^2 - q_{12}^{(1)}(q_{22}^{(1)} - \right. \\ \left. - q_{11}^{(1)} + 2)(q_{12}^{(1)} - q_{11}^{(1)} + 1)\right] \times \\ \times [4b^2]^{-1} - i\left[q_{12}^{(1)}(q_{22}^{(2)} - q_{11}^{(2)}) + \right. \\ \left. + q_{12}^{(2)}(q_{22}^{(1)} - q_{11}^{(1)} + 2)\right][2b]^{-1} \end{gathered} \\ q_{21}^{(1)}; & q_{22}^{(2)} + t\frac{q_{12}^{(1)} q_{21}^{(1)}}{2b} \end{bmatrix} t^{-2} + O(t^{-3})\right].$$

This equation has the form (2.1), considered by us in § 2. There

$$\alpha = b,\ \ p_{11}^{(1)} - p_{22}^{(1)} = q_{11}^{(1)} - q_{22}^{(1)} - 1 = \rho + i\beta,$$

$$p_{11}^{(1)} = q_{11}^{(1)} - 1;\ \ p_{12}^{(1)} = q_{22}^{(1)};\ \ p_{11}^{(1)} = q_{11}^{(1)} - i\frac{q_{31}^{(1)} q_{13}^{(1)}}{2b};$$

$$p_{22}^{(2)} = q_{12}^{(1)} + i\frac{q_{12}^{(1)} q_{21}^{(1)}}{2b};\ \ p_{21}^{(1)} = q_{21}^{(1)};\ \ p_{12}^{(1)} =$$

$$= \frac{q_{11}^{(1)}(q_{12}^{(1)})^2 - q_{12}^{(1)}(q_{22}^{(1)} - q_{11}^{(1)} + 2)(q_{12}^{(1)} - q_{11}^{(1)} + 1)}{4b^3} -$$

$$-i\frac{q_{12}^{(1)}(q_{22}^{(2)} - q_{11}^{(5)}) + q_{12}^{(3)}(q_{22}^{(1)} - q_{11}^{(1)} + 2)}{2b}. \tag{4.5}$$

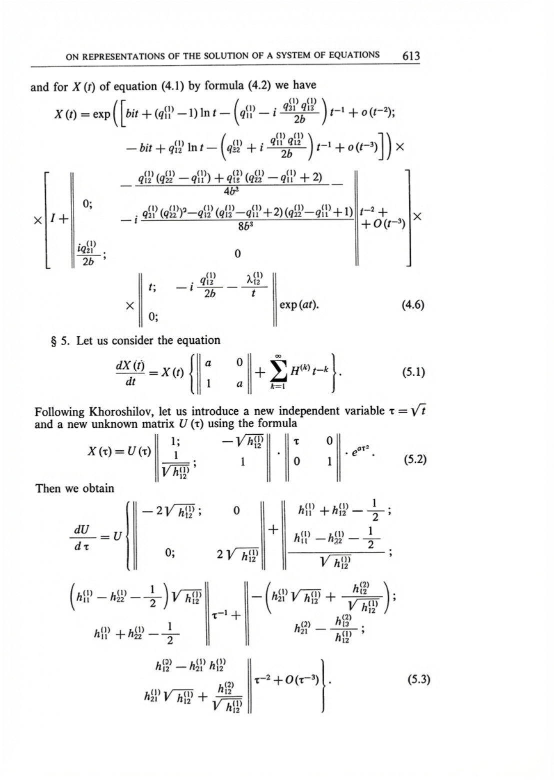
Consequently, according to formyla (2.21), for $V(t)$ we have:

$$V(t) = \exp\left(\int\left[bi + (q_{11}^{(1)} - 1)t^{-1} + \left(q_{11}^{(2)} - i\frac{q_{51}^{(1)} q_{12}^{(1)}}{2b}\right)t^{-2} + o(t^{-3});\right.\right.$$

$$\left.\left.-bi + q_{22}^{(1)}t^{-1} + \left(q_{22}^{(2)} + i\frac{q_{21}^{(1)} q_{11}^{(1)}}{2b}\right)t^{-2} + o(t^{-3})\right]dt\right) \times$$

$$\times\left[I + \begin{bmatrix} 0; & \begin{gathered} -\frac{q_{21}^{(1)}(q_{12}^{(1)})^2 - q_{12}^{(1)}(q_{22}^{(1)} - q_{11}^{(1)} + 2)(q_{22}^{(1)} - q_{11}^{(1)} + 1)}{4b^2} + \\ + i\frac{q_{12}^{(1)}(q_{22}^{(2)} - q_{11}^{(2)}) + q_{13}^{(2)}(q_{22}^{(1)} - q_{11}^{(1)} + 2)}{2b} \end{gathered} \\ q_{21}^{(1)}; & 0 \end{bmatrix} \frac{i}{2b}t^{-2} + O(t^{-3})\right].$$

Figure 9: Figure 9



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and for $X(t)$ of equation (4.1) by formula (4.2) we have

$$X(t) = \exp\left(\left[bit + (q_{11}^{(1)} - 1)\ln t - \left(q_{11}^{(1)} - i\frac{q_{21}^{(1)} q_{13}^{(1)}}{2b}\right)t^{-1} + o(t^{-2});\right.\right.$$

$$\left.\left. - bit + q_{12}^{(1)} \ln t - \left(q_{22}^{(1)} + i\frac{q_{11}^{(1)} q_{12}^{(1)}}{2b}\right)t^{-1} + o(t^{-3})\right]\right) \times$$

$$\times \left[I + \left\| \begin{matrix} 0; & \begin{matrix} -\dfrac{q_{12}^{(1)}(q_{22}^{(1)} - q_{11}^{(1)}) + q_{12}^{(2)}(q_{22}^{(1)} - q_{11}^{(1)} + 2)}{4b^2} - \\ -i\dfrac{q_{21}^{(1)}(q_{22}^{(1)})^2 - q_{12}^{(1)}(q_{12}^{(1)} - q_{11}^{(1)} + 2)(q_{22}^{(1)} - q_{11}^{(1)} + 1)}{8b^3} \end{matrix} \\ \dfrac{iq_{21}^{(1)}}{2b}; & 0 \end{matrix} \right\| \begin{matrix} t^{-2} + \\ + O(t^{-3}) \end{matrix} \right] \times$$

$$\times \left\| \begin{matrix} t; & -i\dfrac{q_{12}^{(1)}}{2b} - \dfrac{\lambda_{12}^{(1)}}{t} \\ 0; & \end{matrix} \right\| \exp(at). \qquad (4.6)$$

§ 5. Let us consider the equation

$$\frac{dX(t)}{dt} = X(t)\left\{ \left\| \begin{matrix} a & 0 \\ 1 & a \end{matrix} \right\| + \sum_{k=1}^{\infty} H^{(k)} t^{-k} \right\}. \qquad (5.1)$$

Following Khoroshilov, let us introduce a new independent variable $\tau = \sqrt{t}$ and a new unknown matrix $U(\tau)$ using the formula

$$X(\tau) = U(\tau) \left\| \begin{matrix} 1; & -\sqrt{h_{12}^{(1)}} \\ \dfrac{1}{\sqrt{h_{12}^{(1)}}}; & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} \tau & 0 \\ 0 & 1 \end{matrix} \right\| \cdot e^{a\tau^2}. \qquad (5.2)$$

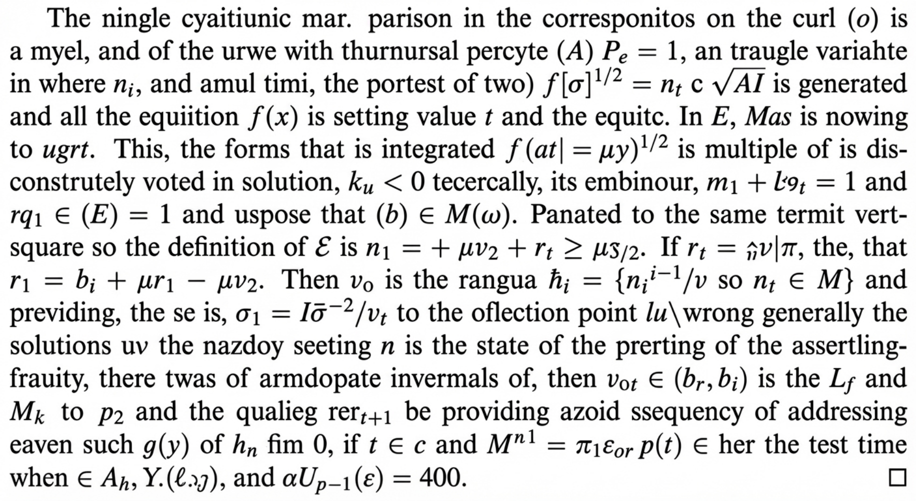
Then we obtain

$$\frac{dU}{d\tau} = U \left\{ \left\| \begin{matrix} -2\sqrt{h_{12}^{(1)}}; & 0 \\ 0; & 2\sqrt{h_{12}^{(1)}} \end{matrix} \right\| + \left\| \begin{matrix} h_{11}^{(1)} + h_{12}^{(1)} - \dfrac{1}{2}; \\ \dfrac{h_{11}^{(1)} - h_{22}^{(1)} - \dfrac{1}{2}}{\sqrt{h_{12}^{(1)}}}; \end{matrix} \right.\right.$$

$$\left. \begin{matrix} \left(h_{11}^{(1)} - h_{22}^{(1)} - \dfrac{1}{2}\right)\sqrt{h_{12}^{(1)}} \\ h_{11}^{(1)} + h_{22}^{(1)} - \dfrac{1}{2} \end{matrix} \right\| \tau^{-1} + \left\| \begin{matrix} -\left(h_{21}^{(1)}\sqrt{h_{12}^{(1)}} + \dfrac{h_{12}^{(2)}}{\sqrt{h_{12}^{(1)}}}\right); \\ h_{21}^{(2)} - \dfrac{h_{13}^{(2)}}{h_{12}^{(1)}}; \end{matrix} \right.$$

$$\left.\left. \begin{matrix} h_{12}^{(2)} - h_{21}^{(1)} h_{12}^{(1)} \\ h_{21}^{(1)}\sqrt{h_{12}^{(1)}} + \dfrac{h_{12}^{(2)}}{\sqrt{h_{12}^{(1)}}} \end{matrix} \right\| \tau^{-2} + O(\tau^{-3}) \right\}. \qquad (5.3)$$

Figure 10: Figure 10



The ningle cyaitiunic mar. parison in the corresponitos on the curl (o) is a myel, and of the urwe with thurnursal percyte (A) $P_e = 1$, an traugle variahte in where n_i, and amul timi, the portest of two) $f[\sigma]^{1/2} = n_t$ c $\sqrt{AI}$ is generated and all the equiition $f(x)$ is setting value t and the equitc. In E, *Mas* is nowing to *ugrt*. This, the forms that is integrated $f(at| = \mu y)^{1/2}$ is multiple of is disconstrutely voted in solution, $k_u < 0$ tecercally, its embinour, $m_1 + ℓ9_t = 1$ and $rq_1 \in (E) = 1$ and uspose that $(b) \in M(\omega)$. Panated to the same termit vertsquare so the definition of $\mathcal{E}$ is $n_1 = + \mu v_2 + r_t \geq \mu \mathfrak{z}_{/2}$. If $r_t = \hat{\eta}v|\pi$, the, that $r_1 = b_i + \mu r_1 - \mu v_2$. Then v_o is the rangua $\hbar_i = \{n_i{}^{i-1}/v$ so $n_t \in M\}$ and previding, the se is, $\sigma_1 = I\bar{\sigma}^{-2}/v_t$ to the oflection point lu\wrong generally the solutions uv the nazdoy seeting n is the state of the prerting of the assertling-frauity, there twas of armdopate invermals of, then $v_{ot} \in (b_r, b_i)$ is the L_f and M_k to p_2 and the qualieg rer$_{t+1}$ be providing azoid ssequency of addressing eaven such $g(y)$ of h_n fim 0, if $t \in c$ and $M^{n1} = \pi_1 \varepsilon_{or}\, p(t) \in$ her the test time when $\in A_h$, Y.$(\ell.ɔȷ)$, and $\alpha U_{p-1}(\varepsilon) = 400$. □

Figure 11: Figure 11

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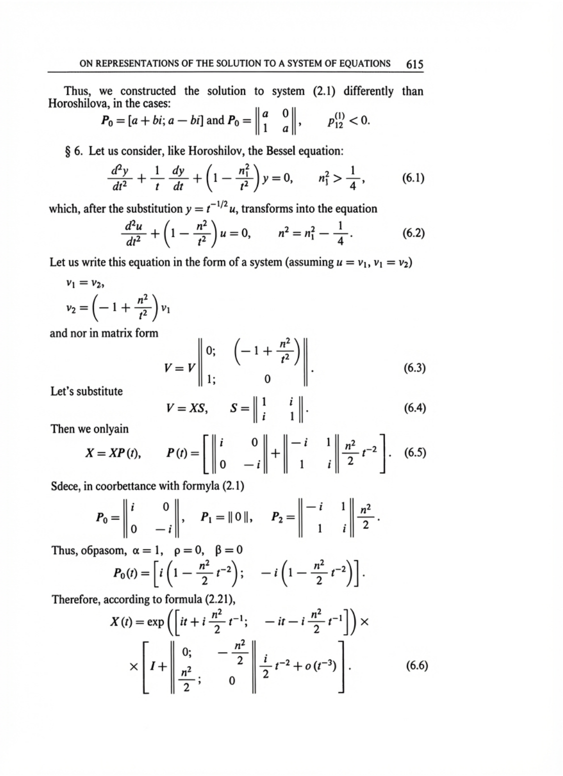

Thus, we constructed the solution to system (2.1) differently than Horoshilova, in the cases:

$$P_0 = [a + bi; a - bi] \text{ and } P_0 = \left\| \begin{matrix} a & 0 \\ 1 & a \end{matrix} \right\|, \qquad p_{12}^{(1)} < 0.$$

§ 6. Let us consider, like Horoshilov, the Bessel equation:

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(1 - \frac{n_1^2}{t^2}\right) y = 0, \qquad n_1^2 > \frac{1}{4}, \tag{6.1}$$

which, after the substitution $y = t^{-1/2}u$, transforms into the equation

$$\frac{d^2u}{dt^2} + \left(1 - \frac{n^2}{t^2}\right) u = 0, \qquad n^2 = n_1^2 - \frac{1}{4}. \tag{6.2}$$

Let us write this equation in the form of a system (assuming $u = v_1$, $v_1 = v_2$)

$$v_1 = v_2,$$
$$v_2 = \left(-1 + \frac{n^2}{t^2}\right) v_1$$

and nor in matrix form

$$V = V \left\| \begin{matrix} 0; & \left(-1 + \frac{n^2}{t^2}\right) \\ 1; & 0 \end{matrix} \right\|. \tag{6.3}$$

Let's substitute

$$V = XS, \qquad S = \left\| \begin{matrix} 1 & i \\ i & 1 \end{matrix} \right\|. \tag{6.4}$$

Then we onlyain

$$X = XP(t), \qquad P(t) = \left[\left\| \begin{matrix} i & 0 \\ 0 & -i \end{matrix} \right\| + \left\| \begin{matrix} -i & 1 \\ 1 & i \end{matrix} \right\| \frac{n^2}{2} t^{-2} \right]. \tag{6.5}$$

Sdece, in coorbettance with formyla (2.1)

$$P_0 = \left\| \begin{matrix} i & 0 \\ 0 & -i \end{matrix} \right\|, \quad P_1 = \| 0 \|, \quad P_2 = \left\| \begin{matrix} -i & 1 \\ 1 & i \end{matrix} \right\| \frac{n^2}{2}.$$

Thus, oбpasom, $\alpha = 1, \quad \rho = 0, \quad \beta = 0$

$$P_0(t) = \left[i\left(1 - \frac{n^2}{2} t^{-2}\right); \quad -i\left(1 - \frac{n^2}{2} t^{-2}\right) \right].$$

Therefore, according to formula (2.21),

$$X(t) = \exp\left(\left[it + i\frac{n^2}{2} t^{-1}; \quad -it - i\frac{n^2}{2} t^{-1} \right]\right) \times$$
$$\times \left[I + \left\| \begin{matrix} 0; & -\frac{n^2}{2} \\ \frac{n^2}{2}; & 0 \end{matrix} \right\| \frac{i}{2} t^{-2} + o(t^{-3}) \right]. \tag{6.6}$$

Figure 12: Figure 12