ON AN OPERATOR SCHEME AND THE SOLVABILITY OF A NUMBER OF QUASILINEAR EQUATIONS OF MECHANICS

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Abstract

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ON AN OPERATOR SCHEME AND THE SOLVABILITY OF A NUMBER OF QUASI-LINEAR EQUATIONS OF MECHANICS

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The paper proves theorems on the solvability of a certain class of nonlinear equations in Banach spaces, making it possible to consider from a general point of view questions of the existence of solutions of quasilinear equations and systems occurring in mechanics.

§ 1. **A general scheme for stationary (elliptic) equations.** Let V, X, Y be a triple of Banach spaces; A, B nonlinear operators. The interrelation of these spaces and operators is expressed by the following diagram

$$\begin{array}{ccccccc} & \xrightarrow{i} & X & \xrightarrow{A} & X^* & & \\ V & & & & & \xrightarrow{i} & V^*, \\ & \xrightarrow{i} & X & \xrightarrow{B} & Y & & \end{array}$$

where i is the embedding operator; V^*, X^* are the conjugate spaces.

Assume that X is separable and that the embedding $V \subset X$ is dense everywhere. Consider the equation

$$A(u) + B(u) = h, \tag{1}$$

where $u \in X$ is the unknown solution, and $h \in X^*$ is given.

Definition. A solution of equation (1) is an element $u \in X$ such that for every $v \in V$ the identity

$$\langle A(u), v\rangle + \langle B(u), v\rangle = \langle h, v\rangle \tag{2}$$

holds.

(Here $\langle w, v\rangle$ is the value of $w \in V^*$ on $v \in V$.)

Assumptions. I. **Ellipticity of $A(u)$.** For every $u \in V$

$$\lim_{\|u\|_X \to \infty} \frac{\langle A(u), u\rangle}{\|u\|_X} = +\infty.$$

II. **Orthogonality of $B(u)$.** For every $u \in V$, $\langle B(u), u\rangle = 0$.

III. The operators A and B are weakly continuous as operators from X into V^*, i.e., if $u_n \to u$ weakly in X, then $\langle A(u_n), v\rangle \to \langle A(u), v\rangle$ and $\langle B(u_n), v\rangle \to \langle B(u), v\rangle$ for every $v \in V$.

Theorem 1. *If conditions I-III are satisfied, then for every $h \in X^*$ equation (1) has at least one solution in the sense of (2).*

The proof of the theorem is carried out by B. G. Galerkin' s method; here the solvability of the moment equations and the a priori estimate follow from conditions I, II, while the possibility of passage to the limit follows from condition III.

§ 2. **Applications to stationary problems.**

1. **The Navier-Stokes system of equations.** Let $G \subset \mathbb{R}^n$, $u = (u_1, \dots, u_n)$,

$$-\Delta u + \sum_{k=1}^{n} u_k \frac{\partial u}{\partial x_k} + \operatorname{grad} p = h(x), \qquad \operatorname{div} u = 0, \qquad u\big|_{\partial G} = 0. \tag{3}$$

We apply Theorem 1 to prove solvability of system (3); accordingly, set

$$X = \{u(x) \mid u(x) \in \dot{W}_2^{(1)},\ \operatorname{div} u = 0\},$$

$$V = \{u(x) \mid u(x) \in \dot{W}_p^{(1)},\ p > n,\ \operatorname{div} u = 0\}, \quad Y = L_1, \quad A(u) = -\Delta u,$$

$$B(u) = \sum_{k=1}^{n} u_k \frac{\partial u}{\partial x_k}.$$

Theorem 2. If $h(x) \in X^*$, then system (3) is solvable.

This result is known; see, for example, ([1]).

Remark. It is not difficult to show that $X^* = W_2^{(-1)}/\Pi$, where Π is the subspace of potentials of the form $u(x) = \operatorname{grad} p(x)$, $p(x) \in L_2$ (cf. ([1])).

2. The system of large deflection of an A. Föppl plate ([2])

$$\Delta^2\xi + b_1(\chi,\xi) = \Delta^2\xi + \frac{\partial^2\chi}{\partial y^2}\frac{\partial^2\xi}{\partial x^2} - 2\frac{\partial^2\chi}{\partial x\partial y}\frac{\partial^2\xi}{\partial x\partial y} + \frac{\partial^2\chi}{\partial x^2}\frac{\partial^2\xi}{\partial y^2} = P(x,y),$$

$$2\Delta^2\chi + b_2(\xi) = 2\Delta^2\chi - 2\frac{\partial^2\xi}{\partial x^2}\frac{\partial^2\xi}{\partial y^2} + 2\left(\frac{\partial^2\xi}{\partial x\partial y}\right)^2 = 0.$$

In the domain $G \subset \mathbf{R}^2$ it is required to find a solution of the first boundary-value problem. Set

$$X = \{u = (\xi,\chi) \mid u \in \dot{W}_2^{(2)}\}, \quad Y = L_1, \quad V = \dot{W}_2^{(2)}, \quad A(u) = (\Delta^2\xi, 2\Delta^2\chi), \quad B(u) = (b_1(\xi,\chi), b_2(\xi)).$$

Theorem 3 (see also ([3],[4])). If $P(x,y) \in W_2^{(-2)}$, then the system of A. Föppl equations is solvable in the space $\dot{W}_2^{(2)}$.

Remark. From the smoothness theorem for the biharmonic equation it follows that if $P(x,y)$ is smooth, then the solution is also smooth.

§ 3. General scheme for nonstationary equations

Notation:

$$L_p(X) = \left\{ u(t) : [0,T] \to X \mid \|u\|^p = \int_0^T \|u\|_X^p \, dt < \infty \right\}, \quad 1 \le p < \infty.$$

Consider the Cauchy problem for the equation

$$u' + A(u) + B(u) = h(t), \quad u(0) = 0, \tag{4}$$

where the operators A and B act in the following scheme:

$$\begin{array}{ccccc} & \xrightarrow{i} & L_p(X) & \xrightarrow{A} & L_{p'}(X^*) \\ L_\infty(V) & & & & \downarrow i \\ & & & & L_1(V^*), \quad p' = \dfrac{p}{p-1}. \\ & \xrightarrow{i} & L_p(X) & \xrightarrow{B} & Y \\ & & & & \uparrow i \end{array}$$

Definitions:

$$F_1 = \{u(t) \mid u \in L_\infty(V),\ u' \in L_1(V^*),\ u(0) = 0\},$$

$$F_2 = \{u(t) \mid u \in L_p(X),\ u' \in L_1(V^*),\ u(0) = 0\}; \quad [w, v] = \int_0^T \langle w, v\rangle\, dt,$$

$$w \in L_1(V^*), \quad v \in L_\infty(V).$$

Assumptions. I. **Parabolicity.** $u' + A(u)$. For any function $u(t) \in F_1$,

$$\lim_{\|u\| \to \infty} \frac{[u' + A(u), u]}{\|u\|} = +\infty.$$

II. **Orthogonality of** $B(u)$**.** If $u(t) \in F_1$, then

$$[B(u), u] = 0.$$

III. The operators A and B are weakly continuous from F_2 into $L_1(V^*)$.

Theorem 4. If conditions I-III are fulfilled, then for every function $h(t) \in L_{p'}(X^*)$ there exists at least one solution $u(t) \in F_2$ of equation (4), in the sense of the identity

$$[u' + A(u) + B(u), v] = [h, v], \quad \forall v(t) \in L_\infty(V).$$

The proof is carried out by the Galerkin method with the introduction of a vanishing "viscosity" (see, for example, ([5]) or ([6])).

§ 4. Applications to Nonstationary Problems

1. The Euler equations of motion of a rigid body in the principal axes of inertia have the form

$$I_m \frac{d\Omega_m}{dt} + (I_{m+2} - I_{m+1})\Omega_{m+1}\Omega_{m+2} = K_m(t), \qquad \Omega_m(0) = 0, \tag{5}$$

where m runs through the cyclic permutation of the indices $1, 2, 3$.

Put $G = \cdot$; $X = V = \mathbf{R}^3$; $A \equiv 0$; $B(\vec{\Omega}) = \{(I_{m+2} - I_{m+1})\Omega_{m+1}\Omega_{m+2}\}$, $m = 1, 2, 3$; $p = 2$. Then $L_2(X) = L_2(0, T)$, $L_1(V^*) = L_1(0, T)$; further, $F_1 = F_2 = \{\vec{\Omega}(t) \mid \vec{\Omega}'(t) \in L_1(0, T),\ \vec{\Omega}(0) = 0\}$.

Theorem 5. If $K_m(t)$ are summable functions, then system (5) has at least one solution $\vec{\Omega}(t) \in F_1$.

Remark. Since $F_1 \subset C(0, T)$, for continuous $K_m(t)$ it follows from equations (5) that $\vec{\Omega}'(t)$ is also continuous.

2. **The nonstationary Navier–Stokes system.** Let $Q = G \times [0,T]$, $G \subset \mathbf{R}^n$, $S = \partial G \times [0,T]$;

$$u' - \Delta u + \sum_{k=1}^{n} u_k \frac{\partial u}{\partial x_k} + \operatorname{grad} p = h(x,t), \qquad \operatorname{div} u = 0, \tag{6}$$

$$u(x,0) = 0, \qquad u|_S = 0. \tag{7}$$

Put $X = \{u(x) \mid u \in \mathring{W}_2^{(2)}(G),\ \operatorname{div} u = 0\}$, $V = \{u(x) \mid u \in \mathring{W}_q^{(1)}, q > n,\ \operatorname{div} u = 0\}$; $p = 2$; $Y = L_1(Q)$, $A(u) = -\Delta u$, $B(u) = \sum_{k=1}^{n} u_k \frac{\partial u}{\partial x_k}$. Then

$$F_2 = \{u(x,t) \mid u \in L_2(\mathring{W}_2^{(1)}),\ u' \in L_1(W_q^{(-1)}),\ u(x,0) = 0\}.$$

Theorem 6. If $h(x,t) \in L_2(X^*)$, then system (6), (7) has at least one solution in F_2.

This theorem refines the theorem of E. Hopf (9) (see also (8)); in particular, the finiteness of u' in $L_1(W_q^{(-1)})$ ensures the continuity of $u(x,t)$ in t in the sense of the space F_2.

3. **Analogy with example 2 of § 2.** Consider in the cylinder $Q = G \times [0,T]$, $G \in \mathbf{R}^{2n}$, the first boundary-value problem for the system

$$\xi' + \Delta^2 \xi - \sum_{i=1}^{n} b_1^i(\xi, \chi) = P_1(x_1, y_1, \dots, x_n, y_n, t),$$

$$\chi' + \Delta^2 \chi - \sum_{i=1}^{n} b_2^i(\xi) = P_2(x_1, y_1, \dots, x_n, y_n, t), \tag{8}$$

where $b_1^i(\xi,\chi)$, $b_2^i(\xi)$ are the same as in § 2, with (x,y) replaced by (x_i, y_i).

Putting $X = \mathring{W}_2^{(2)}(G)$, $V = \mathring{W}_p^{(2)}$, $p > n/2$, $Y = L_1(Q)$, we obtain:

Theorem 7. If $P_i(\dots) \in L_2(W_2^{(-2)})$, $i = 1, 2$, then system (8) is solvable in the space $F_2 = \{u = (\xi, \chi) \mid u \in L_2(\mathring{W}_2^{(2)}),\ u' \in L_1(W_p^{(-2)}),\ u(0) = 0\}$.

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Note: Figure translations are in progress. See original paper for figures.