The minimization of a convex functional by the free trajectories of a linear system

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Abstract

The paper considers the problem of minimizing a convex functional

$$J = \int_{t_0}^{t_0+T} F(x,t)\,dt$$

along the trajectories of a linear control system

$$\frac{dx}{dt} = A(t)x + B(t)u,$$

with both endpoints being free. It is proved that the Pontryagin maximum principle provides a sufficient optimality condition for the control and trajectory in this problem. To find the optimal trajectory and optimal control, a convergent method of successive approximations is proposed. An application to the minimization of a quadratic functional is provided, and a specific example of the "road profile determination" problem is considered. 2 illustrations. 8 bibliographical references.

Full Text

Preamble

This section addresses the optimization of a linear control system described by the following differential equation:

$$\dot{x} = A(t)x + B(t)u \tag{1.1}$$

where $x(t) = (x_1(t), \dots, x_n(t))$ is an n-dimensional state vector and $u(t) = (u_1(t), \dots, u_r(t))$ is an r-dimensional control vector. The matrices $A(t) = [a_{ij}(t)]$ and $B(t) = [b_{ki}(t)]$ are defined on the interval $I = [t_0, t_0 + T]$. The control constraints are given by:

$$|u_k(t)| \le 1, \quad k = 1, 2, \dots, r \tag{1.2}$$

The objective is to minimize a functional of the form:

$$J(u) = \int_{t_0}^{t_0+T} F(x,t)dt \tag{1.3}$$

where $F(x,t)$ is a convex function with respect to x. Problems of this type, defined by equations (1.1)-(1.3), have been extensively studied in the literature [?, ?, ?, ?, ?, ?, ?]. Specifically, the existence and uniqueness of solutions for such systems were established in [?]. In this paper, we propose a numerical method for solving the optimal control problem (1.1)-(1.3) based on the maximum principle and iterative refinement.

§ 2. Necessary Conditions for Optimality

According to the Pontryagin Maximum Principle [?], for a control $u(t)$ to be optimal, there must exist a non-zero adjoint vector function $\psi(t) = (\psi_1(t), \dots, \psi_n(t))$ satisfying the following Hamiltonian system:

$$H(\psi, x, u, t) = (\psi, Ax + Bu) - F(x,t) \tag{2.1}$$

The adjoint equations are given by:

$$\dot{\psi} = -A^*(t)\psi + \frac{\partial F}{\partial x} \tag{2.2}$$

where the control $u(t)$ is chosen to maximize the Hamiltonian:

$$H(\psi(t), x(t), u(t), t) = \max_u H(\psi(t), x(t), u, t) \tag{2.3}$$

For the linear system (1.1), the condition (2.3) implies that the optimal control $u(t)$ must satisfy:

$$(\psi(t), B(t)u(t)) = \max_u (\psi(t), B(t)u) \tag{2.4}$$

Given the constraints (1.2), the components of the control are determined by $u_k(t) = \operatorname{sign}[\sum_{i=1}^n \psi_i(t) b_{ik}(t)]$.

Let us define a support function $\phi(x,t)$ such that for any trajectory $x(t)$ and adjoint variable $\psi(t)$, the following relation holds:

$$\phi(x(t), T) = \phi(x(t_0), t_0) + \int_{t_0}^{t_0+T} \left[\frac{\partial \phi}{\partial t} + \left(\frac{\partial \phi}{\partial x}, Ax + Bu \right) \right] dt \tag{2.8}$$

By choosing $\psi(t) = \frac{\partial \phi}{\partial x}$, we can rewrite the functional $J(u)$ in a form that facilitates iterative improvement. Specifically, for a change in control from u to $\bar{u}$, the corresponding change in the functional can be expressed through the Hamiltonian and the convexity properties of $F(x,t)$.

§ 3. Iterative Method and Convergence

We consider an iterative process where, at each step k, we have a control $u_k(t)$ and a corresponding trajectory $x_k(t)$. To find an improved control $u_{k+1}(t)$, we solve the adjoint equation (2.2) using the current state $x_k(t)$. Let $\psi_k(t)$ be the solution to:

$$\dot{\psi}_k = -A^*(t)\psi_k + \frac{\partial F(x_k, t)}{\partial x} \tag{3.1}$$

with the boundary condition $\psi_k(t_0 + T) = 0$. We then determine a candidate control $v_k(t)$ that maximizes the linear part of the Hamiltonian:

$$(\psi_k(t), B(t)v_k(t)) = \max_u(\psi_k(t), B(t)u) \tag{4.1}$$

The new control is then defined as a convex combination:

$$u_{k+1}(t) = \alpha_k u_k(t) + (1 - \alpha_k)v_k(t) \tag{4.2}$$

where the parameter $\alpha_k \in [0, 1]$ is chosen to ensure the maximum decrease in the functional $J(u)$.

The convergence of this sequence $J(u_k)$ to the minimum value is guaranteed by the convexity of $F(x, t)$ and the properties of the linear system. As shown in (4.4)-(4.7), the difference $J(u_k) - J(u_{k+1})$ remains non-negative, and the sequence of trajectories $x_k(t)$ converges to the optimal trajectory $x^*(t)$ in the L_2 norm.

§ 4. Numerical Implementation

In practice, the parameter α_k can be determined by minimizing the functional $J(\alpha u_k + (1 - \alpha)v_k)$ with respect to α. If the condition (4.12) is satisfied:

$$\int_{t_0}^{t_0+T} (\psi_k(t), B(v_k(t) - u_k(t)))dt = 0 \tag{4.12}$$

then the current control $u_k(t)$ satisfies the necessary conditions for optimality. Otherwise, the integral in (4.13) is strictly positive, ensuring that a step can be taken to reduce the functional.

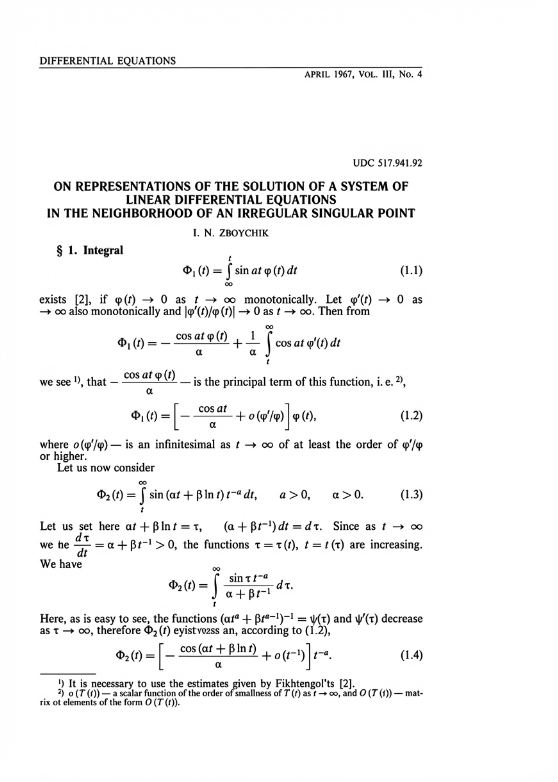
§ 5. Example

Consider the problem of minimizing the distance to a target trajectory $f(t)$:

$$J(u) = \int_0^2 (x(t) - f(t))^2 dt \tag{5.1}$$

subject to $\dot{x} = u$ and $|u| \leq 1$. The adjoint equation is $\dot{\psi} = 2x - 2f$ with $\psi(2) = 0$. Using the iterative method described above, starting from an initial guess $u_0(t) = 0$, the sequence of controls $u_k(t)$ converges to the optimal bang-bang control $u^*(t) = \text{sign}(\psi(t))$. Numerical results demonstrate that the functional decreases monotonically, reaching the optimal value within a few iterations.

Figures



DIFFERENTIAL EQUATIONS
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ON REPRESENTATIONS OF THE SOLUTION OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS IN THE NEIGHBORHOOD OF AN IRREGULAR SINGULAR POINT

I. N. ZBOYCHIK

§ 1. Integral

$$\Phi_1(t) = \int_{\infty}^{t} \sin at\, \varphi(t)\, dt \tag{1.1}$$

exists [2], if $\varphi(t) \to 0$ as $t \to \infty$ monotonically. Let $\varphi'(t) \to 0$ as $\to \infty$ also monotonically and $|\varphi'(t)/\varphi(t)| \to 0$ as $t \to \infty$. Then from

$$\Phi_1(t) = -\frac{\cos at\, \varphi(t)}{\alpha} + \frac{1}{\alpha} \int_{t}^{\infty} \cos at\, \varphi'(t)\, dt$$

we see 1), that $-\frac{\cos at\, \varphi(t)}{\alpha}$ — is the principal term of this function, i. e. 2),

$$\Phi_1(t) = \left[-\frac{\cos at}{\alpha} + o(\varphi'/\varphi)\right] \varphi(t), \tag{1.2}$$

where $o(\varphi'/\varphi)$ — is an infinitesimal as $t \to \infty$ of at least the order of φ'/φ or higher.

Let us now consider

$$\Phi_2(t) = \int_{t}^{\infty} \sin(\alpha t + \beta \ln t)\, t^{-a}\, dt, \qquad a > 0, \qquad \alpha > 0. \tag{1.3}$$

Let us set here $\alpha t + \beta \ln t = \tau$, $(\alpha + \beta t^{-1})\, dt = d\tau$. Since as $t \to \infty$ we he $\frac{d\tau}{dt} = \alpha + \beta t^{-1} > 0$, the functions $\tau = \tau(t)$, $t = t(\tau)$ are increasing. We have

$$\Phi_2(t) = \int_{t}^{\infty} \frac{\sin \tau\, t^{-a}}{\alpha + \beta t^{-1}}\, d\tau.$$

Here, as is easy to see, the functions $(\alpha t^{a} + \beta t^{a-1})^{-1} = \psi(\tau)$ and $\psi'(\tau)$ decrease as $\tau \to \infty$, therefore $\Phi_2(t)$ eyistvozss an, according to (1.2),

$$\Phi_2(t) = \left[-\frac{\cos(\alpha t + \beta \ln t)}{\alpha} + o(t^{-1})\right] t^{-a}. \tag{1.4}$$

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1) It is necessary to use the estimates given by Fikhtengol'ts [2].

2) $o(T(t))$ — a scalar function of the order of smallness of $T(t)$ as $t \to \infty$, and $O(T(t))$ — matrix ot elements of the form $O(T(t))$.

Figure 1: Figure 1

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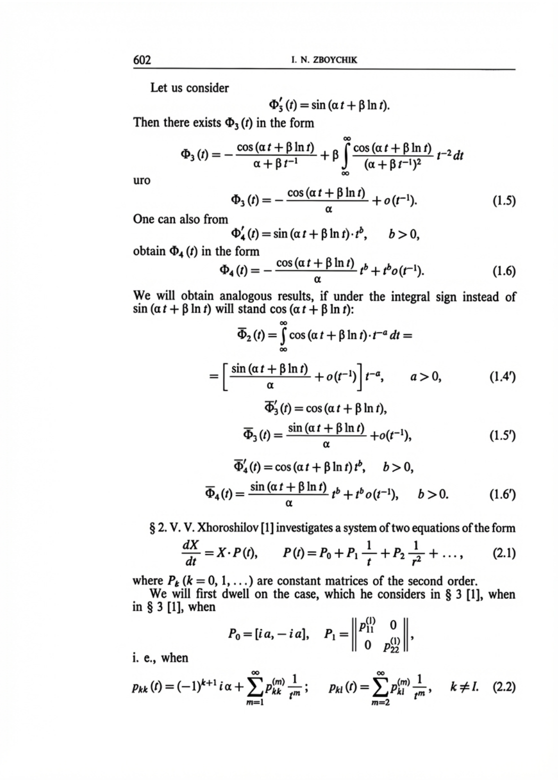


Let us consider

$$\Phi_3'(t) = \sin(\alpha t + \beta \ln t).$$

Then there exists $\Phi_3(t)$ in the form

$$\Phi_3(t) = -\frac{\cos(\alpha t + \beta \ln t)}{\alpha + \beta t^{-1}} + \beta \int_{\infty}^{\infty} \frac{\cos(\alpha t + \beta \ln t)}{(\alpha + \beta t^{-1})^2} t^{-2}\, dt$$

uro

$$\Phi_3(t) = -\frac{\cos(\alpha t + \beta \ln t)}{\alpha} + o(t^{-1}). \tag{1.5}$$

One can also from

$$\Phi_4'(t) = \sin(\alpha t + \beta \ln t) \cdot t^b, \qquad b > 0,$$

obtain $\Phi_4(t)$ in the form

$$\Phi_4(t) = -\frac{\cos(\alpha t + \beta \ln t)}{\alpha} t^b + t^b o(t^{-1}). \tag{1.6}$$

We will obtain analogous results, if under the integral sign instead of $\sin(\alpha t + \beta \ln t)$ will stand $\cos(\alpha t + \beta \ln t)$:

$$\overline{\Phi}_2(t) = \int_{\infty}^{\infty} \cos(\alpha t + \beta \ln t) \cdot t^{-a}\, dt =$$

$$= \left[\frac{\sin(\alpha t + \beta \ln t)}{\alpha} + o(t^{-1})\right] t^{-a}, \qquad a > 0, \tag{1.4'}$$

$$\overline{\Phi}_3'(t) = \cos(\alpha t + \beta \ln t),$$

$$\overline{\Phi}_3(t) = \frac{\sin(\alpha t + \beta \ln t)}{\alpha} + o(t^{-1}), \tag{1.5'}$$

$$\overline{\Phi}_4'(t) = \cos(\alpha t + \beta \ln t)\, t^b, \qquad b > 0,$$

$$\overline{\Phi}_4(t) = \frac{\sin(\alpha t + \beta \ln t)}{\alpha} t^b + t^b o(t^{-1}), \qquad b > 0. \tag{1.6'}$$

§ 2. V. V. Xhoroshilov [1] investigates a system of two equations of the form

$$\frac{dX}{dt} = X \cdot P(t), \qquad P(t) = P_0 + P_1 \frac{1}{t} + P_2 \frac{1}{t^2} + \dots, \tag{2.1}$$

where P_k $(k = 0, 1, \dots)$ are constant matrices of the second order.

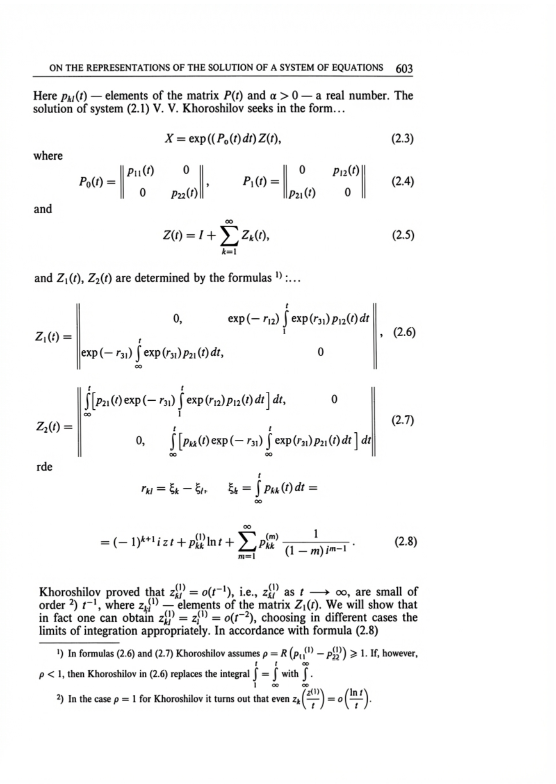
We will first dwell on the case, which he considers in § 3 [1], when in § 3 [1], when

$$P_0 = [i a, -i a], \quad P_1 = \left\| \begin{matrix} p_{11}^{(1)} & 0 \\ 0 & p_{22}^{(1)} \end{matrix} \right\|,$$

i. e., when

$$p_{kk}(t) = (-1)^{k+1} i \alpha + \sum_{m=1}^{\infty} p_{kk}^{(m)} \frac{1}{t^m}; \qquad p_{kl}(t) = \sum_{m=2}^{\infty} p_{kl}^{(m)} \frac{1}{t^m}, \qquad k \neq l. \tag{2.2}$$

Figure 2: Figure 2



Here $p_{kl}(t)$ — elements of the matrix $P(t)$ and $\alpha > 0$ — a real number. The solution of system (2.1) V. V. Khoroshilov seeks in the form…

$$X = \exp((P_o(t)\,dt)\,Z(t), \tag{2.3}$$

where

$$P_0(t) = \left\| \begin{matrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{matrix} \right\|, \qquad P_1(t) = \left\| \begin{matrix} 0 & p_{12}(t) \\ p_{21}(t) & 0 \end{matrix} \right\| \tag{2.4}$$

and

$$Z(t) = I + \sum_{k=1}^{\infty} Z_k(t), \tag{2.5}$$

and $Z_1(t)$, $Z_2(t)$ are determined by the formulas [1] :…

$$Z_1(t) = \left\| \begin{matrix} 0, & \exp(-r_{12}) \int\limits_1^t \exp(r_{31})\, p_{12}(t)\, dt \\ \exp(-r_{31}) \int\limits_\infty^t \exp(r_{31})\, p_{21}(t)\, dt, & 0 \end{matrix} \right\|, \tag{2.6}$$

$$Z_2(t) = \left\| \begin{matrix} \int\limits_\infty^t \left[p_{21}(t) \exp(-r_{31}) \int\limits_1^t \exp(r_{12})\, p_{12}(t)\, dt \right] dt, & 0 \\ 0, & \int\limits_\infty^t \left[p_{kk}(t) \exp(-r_{31}) \int\limits_\infty^t \exp(r_{31})\, p_{21}(t)\, dt \right] dt \end{matrix} \right\| \tag{2.7}$$

rde

$$r_{kl} = \xi_k - \xi_l, \qquad \xi_k = \int\limits_\infty^t p_{kk}(t)\, dt =$$

$$= (-1)^{k+1}\, i\, z\, t + p_{kk}^{(1)} \ln t + \sum_{m=1}^{\infty} p_{kk}^{(m)} \frac{1}{(1-m)\, i^{m-1}}\,. \tag{2.8}$$

Khoroshilov proved that $z_{kl}^{(1)} = o(t^{-1})$, i.e., $z_{kl}^{(1)}$ as $t \longrightarrow \infty$, are small of order [2] t^{-1}, where $z_{kl}^{(1)}$ — elements of the matrix $Z_1(t)$. We will show that in fact one can obtain $z_{kl}^{(1)} = z_l^{(1)} = o(t^{-2})$, choosing in different cases the limits of integration appropriately. In accordance with formula (2.8)

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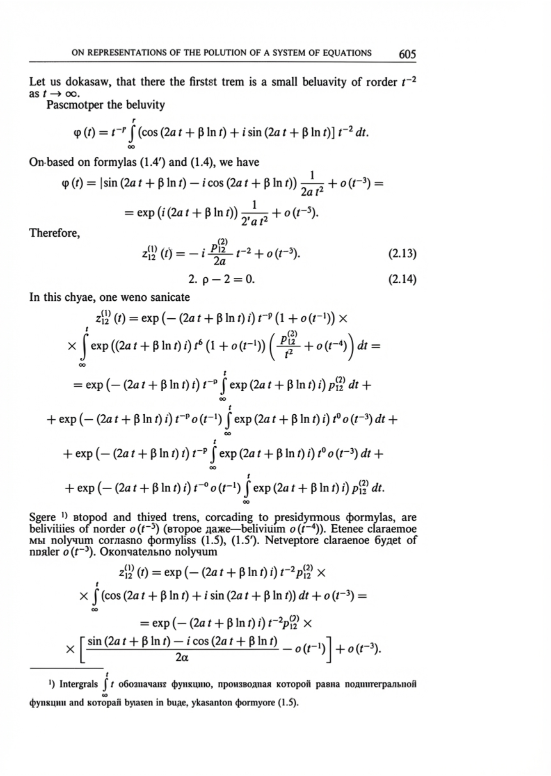
[1] In formulas (2.6) and (2.7) Khoroshilov assumes $\rho = R\left(p_{11}^{(1)} - p_{22}^{(1)}\right) \geqslant 1$. If, however, $\rho < 1$, then Khoroshilov in (2.6) replaces the integral $\int\limits_1^t = \int\limits_\infty^t$ with $\int\limits_\infty^\infty$.

[2] In the case $\rho = 1$ for Khoroshilov it turns out that even $z_k\left(\frac{z^{(1)}}{t}\right) = o\left(\frac{\ln t}{t}\right)$.

Figure 3: Figure 3



ON REPRESENTATIONS OF THE POLUTION OF A SYSTEM OF EQUATIONS 605

Let us dokasaw, that there the firstst trem is a small beluavity of rorder t^{-2} as $t \to \infty$.

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$$\varphi(t) = t^{-r} \int_{\infty}^{r} (\cos(2at + \beta \ln t) + i \sin(2at + \beta \ln t)]\, t^{-2}\, dt.$$

On-based on formylas (1.4′) and (1.4), we have

$$\varphi(t) = |\sin(2at + \beta \ln t) - i \cos(2at + \beta \ln t))\, \frac{1}{2at^2} + o(t^{-3}) =$$

$$= \exp(i(2at + \beta \ln t))\, \frac{1}{2'at^2} + o(t^{-5}).$$

Therefore,

$$z_{12}^{(1)}(t) = -i\, \frac{p_{12}^{(2)}}{2a}\, t^{-2} + o(t^{-3}). \tag{2.13}$$

$$2.\ \rho - 2 = 0. \tag{2.14}$$

In this chyae, one weno sanicate

$$z_{12}^{(1)}(t) = \exp(-(2at + \beta \ln t)\, i)\, t^{-\rho}\, (1 + o(t^{-1})) \times$$

$$\times \int_{\infty}^{t} \exp((2at + \beta \ln t)\, i)\, t^{6}\, (1 + o(t^{-1})) \left(\frac{p_{12}^{(2)}}{t^2} + o(t^{-4}) \right) dt =$$

$$= \exp(-(2at + \beta \ln t)\, t)\, t^{-\rho} \int_{\infty}^{t} \exp(2at + \beta \ln t)\, i)\, p_{12}^{(2)}\, dt +$$

$$+ \exp(-(2at + \beta \ln t)\, i)\, t^{-\rho}\, o(t^{-1}) \int_{\infty}^{t} \exp(2at + \beta \ln t)\, i)\, t^{0}\, o(t^{-3})\, dt +$$

$$+ \exp(-(2at + \beta \ln t)\, t)\, t^{-\rho} \int_{\infty}^{t} \exp(2at + \beta \ln t)\, i)\, t^{0}\, o(t^{-3})\, dt +$$

$$+ \exp(-(2at + \beta \ln t)\, i)\, t^{-0}\, o(t^{-1}) \int_{\infty}^{t} \exp(2at + \beta \ln t)\, i)\, p_{12}^{(2)}\, dt.$$

Sgere (1) вtopod and thiзed trens, corcading to presidyrmous фormylas, are belivitiies of norder $o(t^{-5})$ (второе даже—beliviuim $o(t^{-4})$). Etenee claraemoe мы nolyчum corлasno фormyliss (1.5), (1.5′). Netveptore claraenoe будет of nnяler $o(t^{-3})$. Окопчательпо nolyчum

$$z_{12}^{(1)}(t) = \exp(-(2at + \beta \ln t)\, i)\, t^{-2} p_{12}^{(2)} \times$$

$$\times \int_{\infty}^{t} (\cos(2at + \beta \ln t) + i \sin(2at + \beta \ln t))\, dt + o(t^{-3}) =$$

$$= \exp(-(2at + \beta \ln t)\, i)\, t^{-2} p_{12}^{(2)} \times$$

$$\times \left[\frac{\sin(2at + \beta \ln t) - i \cos(2at + \beta \ln t)}{2\alpha} - o(t^{-1}) \right] + o(t^{-3}).$$

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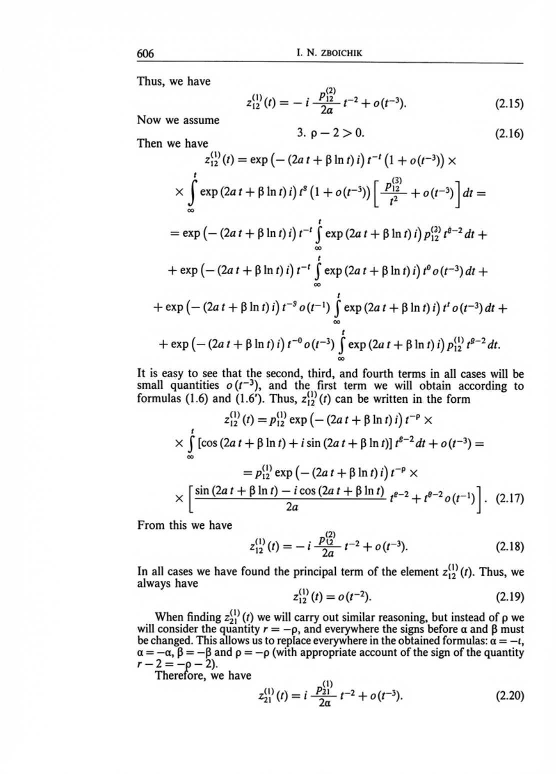
(1) Intergrals $\int_{\infty}^{t} t$ обозпачавг фуикцию, производпая которой равна подиитегральной фупкции and которай byıasen in buде, ykasanton фormyore (1.5).

Figure 4: Figure 4



Thus, we have

$$z_{12}^{(1)}(t) = -i\,\frac{p_{12}^{(2)}}{2a}\,t^{-2} + o(t^{-3}). \tag{2.15}$$

Now we assume

$$3.\ \rho - 2 > 0. \tag{2.16}$$

Then we have

$$\begin{aligned} z_{12}^{(1)}(t) &= \exp\big(-(2a\,t + \beta\ln t)\,i\big)\,t^{-t}\big(1 + o(t^{-3})\big) \times \\ &\times \int_{\infty}^{t} \exp(2a\,t + \beta\ln t)\,i\big)\,t^{\varrho}\big(1 + o(t^{-3})\big)\left[\frac{p_{12}^{(3)}}{t^2} + o(t^{-3})\right]dt = \\ &= \exp\big(-(2a\,t + \beta\ln t)\,i\big)\,t^{-t}\int_{\infty}^{t} \exp(2a\,t + \beta\ln t)\,i\big)\,p_{12}^{(2)}\,t^{\varrho-2}\,dt + \\ &+ \exp\big(-(2a\,t + \beta\ln t)\,i\big)\,t^{-t}\int_{\infty}^{t} \exp(2a\,t + \beta\ln t)\,i\big)\,t^{0}\,o(t^{-3})\,dt + \\ &+ \exp\big(-(2a\,t + \beta\ln t)\,i\big)\,t^{-9}\,o(t^{-1})\int_{\infty}^{t} \exp(2a\,t + \beta\ln t)\,i\big)\,t^{t}\,o(t^{-3})\,dt + \\ &+ \exp\big(-(2a\,t + \beta\ln t)\,i\big)\,t^{-0}\,o(t^{-3})\int_{\infty}^{t} \exp(2a\,t + \beta\ln t)\,i\big)\,p_{12}^{(1)}\,t^{\varrho-2}\,dt. \end{aligned}$$

It is easy to see that the second, third, and fourth terms in all cases will be small quantities $o(t^{-3})$, and the first term we will obtain according to formulas (1.6) and (1.6′). Thus, $z_{12}^{(1)}(t)$ can be written in the form

$$\begin{aligned} z_{12}^{(1)}(t) &= p_{12}^{(1)}\exp\big(-(2a\,t + \beta\ln t)\,i\big)\,t^{-\rho} \times \\ &\times \int_{\infty}^{t} [\cos(2a\,t + \beta\ln t) + i\sin(2a\,t + \beta\ln t)]\,t^{\varrho-2}\,dt + o(t^{-3}) = \\ &= p_{12}^{(1)}\exp\big(-(2a\,t + \beta\ln t)\,i\big)\,t^{-\rho} \times \\ &\times \left[\frac{\sin(2a\,t + \beta\ln t) - i\cos(2a\,t + \beta\ln t)}{2a}\,t^{\varrho-2} + t^{\theta-2}\,o(t^{-1})\right]. \end{aligned} \tag{2.17}$$

From this we have

$$z_{12}^{(1)}(t) = -i\,\frac{p_{12}^{(2)}}{2a}\,t^{-2} + o(t^{-3}). \tag{2.18}$$

In all cases we have found the principal term of the element $z_{12}^{(1)}(t)$. Thus, we always have

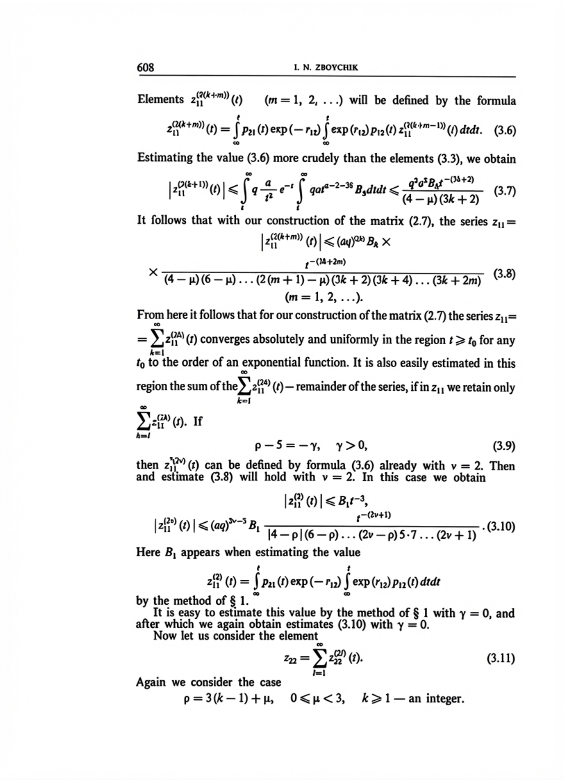
$$z_{12}^{(1)}(t) = o(t^{-2}). \tag{2.19}$$

When finding $z_{21}^{(1)}(t)$ we will carry out similar reasoning, but instead of ρ we will consider the quantity $r = -\rho$, and everywhere the signs before α and β must be changed. This allows us to replace everywhere in the obtained formulas: $\alpha = -\iota$, $\alpha = -\alpha$, $\beta = -\beta$ and $\rho = -\rho$ (with appropriate account of the sign of the quantity $r - 2 = -\rho - 2$).

Therefore, we have

$$z_{21}^{(1)}(t) = i\,\frac{p_{21}^{(1)}}{2\alpha}\,t^{-2} + o(t^{-3}). \tag{2.20}$$

Figure 5: Figure 5



Elements $z_{11}^{(2(k+m))}(t)$ $(m=1,\ 2,\ \ldots)$ will be defined by the formula

$$z_{11}^{(2(k+m))}(t)=\int_{\infty}^{t}p_{21}(t)\exp(-r_{12})\int_{\infty}^{t}\exp(r_{12})p_{12}(t)\,z_{11}^{(2(k+m-1))}(t)\,dtdt. \quad (3.6)$$

Estimating the value (3.6) more crudely than the elements (3.3), we obtain

$$\left|z_{11}^{(2(k+1))}(t)\right|\leqslant\int_{t}^{\infty}q\frac{a}{t^{2}}e^{-t}\int_{t}^{\infty}qat^{a-2-3s}B_{s}dtdt\leqslant\frac{q^{2}a^{s}B_{s}t^{-(3s+2)}}{(4-\mu)(3k+2)} \quad (3.7)$$

It follows that with our construction of the matrix (2.7), the series $z_{11}=$

$$\left|z_{11}^{(2(k+m))}(t)\right|\leqslant(aq)^{(2k)}B_{k}\times$$

$$\times\frac{t^{-(3k+2m)}}{(4-\mu)(6-\mu)\ldots(2(m+1)-\mu)(3k+2)(3k+4)\ldots(3k+2m)} \quad (3.8)$$

$$(m=1,\ 2,\ \ldots).$$

From here it follows that for our construction of the matrix (2.7) the series $z_{11}=$ $=\sum_{k=1}^{\infty}z_{11}^{(2k)}(t)$ converges absolutely and uniformly in the region $t\geqslant t_0$ for any t_0 to the order of an exponential function. It is also easily estimated in this region the sum of the $\sum_{k=1}^{\infty}z_{11}^{(2k)}(t)$ — remainder of the series, if in z_{11} we retain only $\sum_{k=l}^{\infty}z_{11}^{(2k)}(t)$. If

$$\rho-5=-\gamma,\quad \gamma>0, \quad (3.9)$$

then $z_{11}^{(2\nu)}(t)$ can be defined by formula (3.6) already with $\nu=2$. Then and estimate (3.8) will hold with $\nu=2$. In this case we obtain

$$\left|z_{11}^{(2)}(t)\right|\leqslant B_{1}t^{-3},$$

$$\left|z_{11}^{(2\nu)}(t)\right|\leqslant(aq)^{2\nu-5}B_{1}\frac{t^{-(2\nu+1)}}{|4-\rho|(6-\rho)\ldots(2\nu-\rho)5\cdot7\ldots(2\nu+1)}. \quad (3.10)$$

Here B_1 appears when estimating the value

$$z_{11}^{(2)}(t)=\int_{\infty}^{t}p_{21}(t)\exp(-r_{12})\int_{\infty}^{t}\exp(r_{12})p_{12}(t)\,dtdt$$

by the method of § 1.

It is easy to estimate this value by the method of § 1 with $\gamma=0$, and after which we again obtain estimates (3.10) with $\gamma=0$.

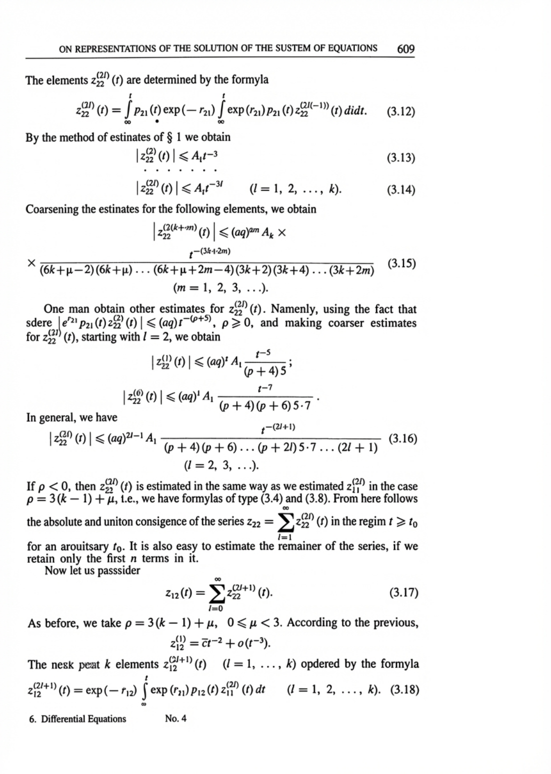
Now let us consider the element

$$z_{22}=\sum_{l=1}^{\infty}z_{22}^{(2l)}(t). \quad (3.11)$$

Again we consider the case

$$\rho=3(k-1)+\mu,\quad 0\leqslant\mu<3,\quad k\geqslant1 \text{ — an integer.}$$

Figure 6: Figure 6



The elements $z_{22}^{(2l)}(t)$ are determined by the formyla

$$z_{22}^{(2l)}(t) = \int_{\infty}^{t} p_{21}(t)\exp(-r_{21})\int_{\infty}^{t}\exp(r_{21})p_{21}(t)z_{22}^{(2l(-1))}(t)\,didt. \quad (3.12)$$

By the method of estinates of § 1 we obtain

$$|z_{22}^{(2)}(t)| \leqslant A_1 t^{-3} \quad (3.13)$$

$$\cdots\cdots\cdots$$

$$|z_{22}^{(2l)}(t)| \leqslant A_l t^{-3l} \qquad (l = 1,\ 2,\ \ldots,\ k). \quad (3.14)$$

Coarsening the estinates for the following elements, we obtain

$$\left|z_{22}^{(2(k+m)}(t)\right| \leqslant (aq)^{am} A_k \times$$

$$\times \frac{t^{-(3k+2m)}}{(6k+\mu-2)(6k+\mu)\ldots(6k+\mu+2m-4)(3k+2)(3k+4)\ldots(3k+2m)} \quad (3.15)$$

$$(m = 1,\ 2,\ 3,\ \ldots).$$

One man obtain other estimates for $z_{22}^{(2l)}(t)$. Namenly, using the fact that sdere $|e^{r_{21}}p_{21}(t)z_{22}^{(2)}(t)| \leqslant (aq)t^{-(\rho+5)}$, $\rho \geqslant 0$, and making coarser estimates for $z_{22}^{(2l)}(t)$, starting with $l = 2$, we obtain

$$|z_{22}^{(1)}(t)| \leqslant (aq)^t A_1 \frac{t^{-5}}{(p+4)5};$$

$$|z_{22}^{(6)}(t)| \leqslant (aq)^1 A_1 \frac{t^{-7}}{(p+4)(p+6)5\cdot 7}.$$

In general, we have

$$|z_{22}^{(2l)}(t)| \leqslant (aq)^{2l-1} A_1 \frac{t^{-(2l+1)}}{(p+4)(p+6)\ldots(p+2l)5\cdot 7\ldots(2l+1)} \quad (3.16)$$

$$(l = 2,\ 3,\ \ldots).$$

If $\rho < 0$, then $z_{22}^{(2l)}(t)$ is estimated in the same way as we estimated $z_{11}^{(2l)}$ in the case $\rho = 3(k-1)+\mu$, t.e., we have formylas of type (3.4) and (3.8). From here follows the absolute and uniton consigence of the series $z_{22} = \sum_{l=1}^{\infty} z_{22}^{(2l)}(t)$ in the regim $t \geqslant t_0$ for an arouitsary t_0. It is also easy to estimate the remainer of the series, if we retain only the first n terms in it.

Now let us passsider

$$z_{12}(t) = \sum_{l=0}^{\infty} z_{22}^{(2l+1)}(t). \quad (3.17)$$

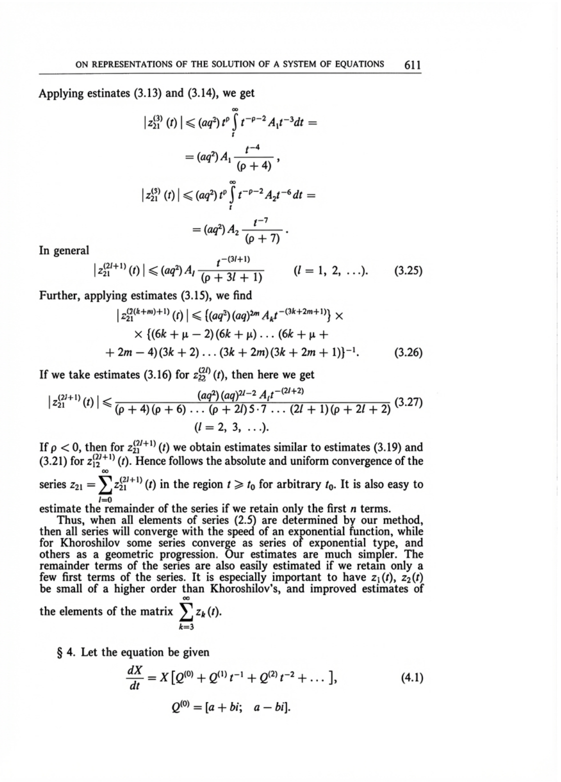
As before, we take $\rho = 3(k-1)+\mu$, $0 \leqslant \mu < 3$. According to the previous,

$$z_{12}^{(1)} = \bar{c}t^{-2} + o(t^{-3}).$$

The nesk peat k elements $z_{12}^{(2l+1)}(t)$ $(l = 1,\ \ldots,\ k)$ opdered by the formyla

$$z_{12}^{(2l+1)}(t) = \exp(-r_{12})\int_{\infty}^{t}\exp(r_{21})p_{12}(t)z_{11}^{(2l)}(t)\,dt \qquad (l = 1,\ 2,\ \ldots,\ k). \quad (3.18)$$

Figure 7: Figure 7



Applying estinates (3.13) and (3.14), we get

$$|z_{21}^{(3)}(t)| \leqslant (aq^2)\, t^{\rho} \int_t^{\infty} t^{-\rho-2} A_1 t^{-3} dt =$$

$$= (aq^2)\, A_1 \frac{t^{-4}}{(\rho+4)},$$

$$|z_{21}^{(5)}(t)| \leqslant (aq^2)\, t^{\rho} \int_t^{\infty} t^{-\rho-2} A_2 t^{-6}\, dt =$$

$$= (aq^2)\, A_2 \frac{t^{-7}}{(\rho+7)}.$$

In general

$$|z_{21}^{(2l+1)}(t)| \leqslant (aq^2)\, A_l \frac{t^{-(3l+1)}}{(\rho+3l+1)} \qquad (l = 1,\ 2,\ \ldots). \tag{3.25}$$

Further, applying estimates (3.15), we find

$$\begin{gathered} |z_{21}^{(2(k+m)+1)}(t)| \leqslant \{(aq^2)(aq)^{2m} A_k t^{-(3k+2m+1)}\} \times \\ \times \{(6k+\mu-2)(6k+\mu)\ldots(6k+\mu+ \\ +2m-4)(3k+2)\ldots(3k+2m)(3k+2m+1)\}^{-1}. \end{gathered} \tag{3.26}$$

If we take estimates (3.16) for $z_{22}^{(2l)}(t)$, then here we get

$$|z_{21}^{(2l+1)}(t)| \leqslant \frac{(aq^2)(aq)^{2l-2} A_l t^{-(2l+2)}}{(\rho+4)(\rho+6)\ldots(\rho+2l)\,5\cdot 7\ldots(2l+1)(\rho+2l+2)} \tag{3.27}$$

$$(l = 2,\ 3,\ \ldots).$$

If $\rho < 0$, then for $z_{21}^{(2l+1)}(t)$ we obtain estimates similar to estimates (3.19) and (3.21) for $z_{12}^{(2l+1)}(t)$. Hence follows the absolute and uniform convergence of the series $z_{21} = \sum_{l=0}^{\infty} z_{21}^{(2l+1)}(t)$ in the region $t \geqslant t_0$ for arbitrary t_0. It is also easy to estimate the remainder of the series if we retain only the first n terms.

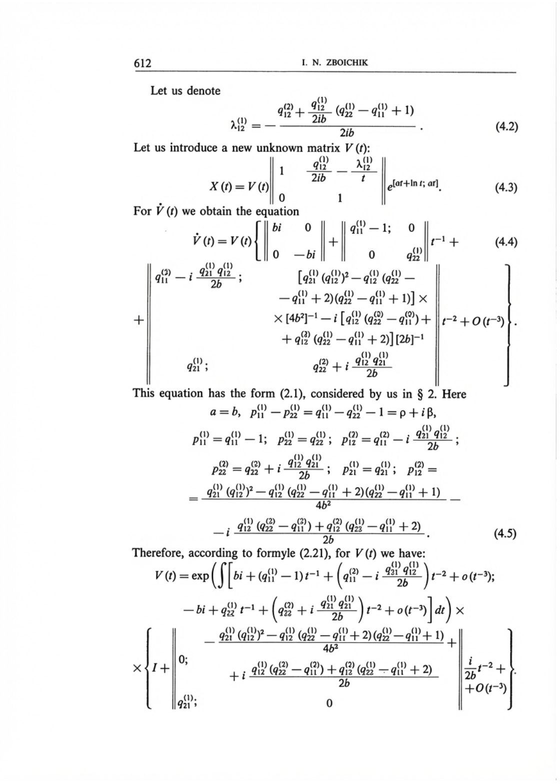
Thus, when all elements of series (2.5) are determined by our method, then all series will converge with the speed of an exponential function, while for Khoroshilov some series converge as series of exponential type, and others as a geometric progression. Our estimates are much simpler. The remainder terms of the series are also easily estimated if we retain only a few first terms of the series. It is especially important to have $z_1(t)$, $z_2(t)$ be small of a higher order than Khoroshilov's, and improved estimates of the elements of the matrix $\sum_{k=3}^{\infty} z_k(t)$.

§ 4. Let the equation be given

$$\frac{dX}{dt} = X\,[Q^{(0)} + Q^{(1)} t^{-1} + Q^{(2)} t^{-2} + \ldots\,], \tag{4.1}$$

$$Q^{(0)} = [a+bi;\quad a-bi].$$

Figure 8: Figure 8



Let us denote

$$\lambda_{12}^{(1)} = -\frac{q_{12}^{(2)} + \frac{q_{12}^{(1)}}{2ib}\,(q_{22}^{(1)} - q_{11}^{(1)} + 1)}{2ib}\,. \tag{4.2}$$

Let us introduce a new unknown matrix $V(t)$:

$$X(t) = V(t)\begin{Vmatrix} 1 & \frac{q_{12}^{(1)}}{2ib} - \frac{\lambda_{12}^{(1)}}{t} \\ 0 & 1 \end{Vmatrix} e^{[at+\ln t;\ at]}. \tag{4.3}$$

For $\dot{V}(t)$ we obtain the equation

$$\dot{V}(t) = V(t)\left\{\begin{Vmatrix} bi & 0 \\ 0 & -bi \end{Vmatrix} + \begin{Vmatrix} q_{11}^{(1)} - 1; & 0 \\ 0 & q_{22}^{(1)} \end{Vmatrix} t^{-1} + \right. \tag{4.4}$$

$$\left. + \begin{Vmatrix} q_{11}^{(2)} - i\,\frac{q_{21}^{(1)}\,q_{12}^{(1)}}{2b}; & \begin{matrix}[q_{21}^{(1)}\,(q_{12}^{(1)})^2 - q_{12}^{(1)}\,(q_{22}^{(1)} - \\ - q_{11}^{(1)} + 2)(q_{22}^{(1)} - q_{11}^{(1)} + 1)] \times \\ \times [4b^2]^{-1} - i\,[q_{12}^{(1)}\,(q_{22}^{(2)} - q_{11}^{(2)}) + \\ + q_{12}^{(2)}\,(q_{22}^{(1)} - q_{11}^{(1)} + 2)]\,[2b]^{-1}\end{matrix} \\ q_{21}^{(1)}; & q_{22}^{(2)} + i\,\frac{q_{12}^{(1)}\,q_{21}^{(1)}}{2b} \end{Vmatrix} t^{-2} + O(t^{-3})\right\}.$$

This equation has the form (2.1), considered by us in § 2. Here

$$a = b,\quad p_{11}^{(1)} - p_{22}^{(1)} = q_{11}^{(1)} - q_{22}^{(1)} - 1 = \rho + i\beta,$$

$$p_{11}^{(1)} = q_{11}^{(1)} - 1;\quad p_{22}^{(1)} = q_{22}^{(1)};\quad p_{12}^{(2)} = q_{11}^{(2)} - i\,\frac{q_{21}^{(1)}\,q_{12}^{(1)}}{2b};$$

$$p_{22}^{(2)} = q_{22}^{(2)} + i\,\frac{q_{12}^{(1)}\,q_{21}^{(1)}}{2b};\quad p_{21}^{(1)} = q_{21}^{(1)};\quad p_{12}^{(2)} =$$

$$= \frac{q_{21}^{(1)}\,(q_{12}^{(1)})^2 - q_{12}^{(1)}\,(q_{22}^{(1)} - q_{11}^{(1)} + 2)(q_{22}^{(1)} - q_{11}^{(1)} + 1)}{4b^2} -$$

$$- i\,\frac{q_{12}^{(1)}\,(q_{22}^{(2)} - q_{11}^{(2)}) + q_{12}^{(2)}\,(q_{23}^{(1)} - q_{11}^{(1)} + 2)}{2b}\,. \tag{4.5}$$

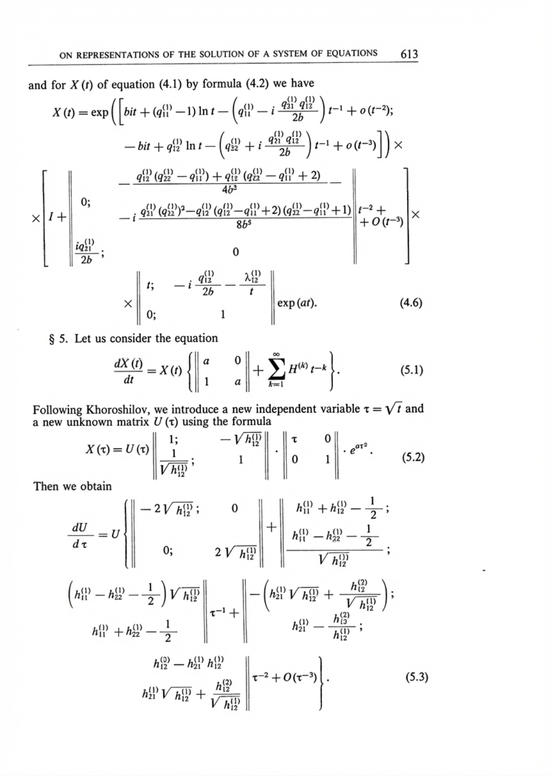
Therefore, according to formyle (2.21), for $V(t)$ we have:

$$V(t) = \exp\left(\int\left[bi + (q_{11}^{(1)} - 1)\,t^{-1} + \left(q_{11}^{(2)} - i\,\frac{q_{21}^{(1)}\,q_{12}^{(1)}}{2b}\right)t^{-2} + o(t^{-3});\right.\right.$$

$$\left.\left. - bi + q_{22}^{(1)}\,t^{-1} + \left(q_{22}^{(2)} + i\,\frac{q_{21}^{(1)}\,q_{21}^{(1)}}{2b}\right)t^{-2} + o(t^{-3})\right]dt\right) \times$$

$$\times\left\{I + \begin{Vmatrix} 0; & \begin{matrix} -\frac{q_{21}^{(1)}\,(q_{12}^{(1)})^2 - q_{12}^{(1)}\,(q_{22}^{(1)} - q_{11}^{(1)} + 2)\,(q_{22}^{(1)} - q_{11}^{(1)} + 1)}{4b^2} + \\ + i\,\frac{q_{12}^{(1)}\,(q_{22}^{(2)} - q_{11}^{(2)}) + q_{12}^{(2)}\,(q_{22}^{(1)} - q_{11}^{(1)} + 2)}{2b} \end{matrix} \\ q_{21}^{(1)}; & 0 \end{Vmatrix} \begin{matrix}\frac{i}{2b}t^{-2} + \\ + O(t^{-3})\end{matrix}\right\}.$$

Figure 9: Figure 9



and for $X(t)$ of equation (4.1) by formula (4.2) we have

$$X(t)=\exp\left(\left[bit+(q_{11}^{(1)}-1)\ln t-\left(q_{11}^{(1)}-i\frac{q_{31}^{(1)}q_{12}^{(1)}}{2b}\right)t^{-1}+o(t^{-2});\right.\right.$$
$$\left.\left.-bit+q_{12}^{(1)}\ln t-\left(q_{22}^{(1)}+i\frac{q_{21}^{(1)}q_{12}^{(1)}}{2b}\right)t^{-1}+o(t^{-3})\right]\right)\times$$
$$\times\left[I+\left\|\begin{matrix}0; & \begin{matrix}-\dfrac{q_{12}^{(1)}(q_{22}^{(1)}-q_{11}^{(1)})+q_{12}^{(1)}(q_{22}^{(1)}-q_{11}^{(1)}+2)}{4b^3}-\\ -i\dfrac{q_{21}^{(1)}(q_{22}^{(1)})^2-q_{12}^{(1)}(q_{12}^{(1)}-q_{11}^{(1)}+2)(q_{22}^{(1)}-q_{11}^{(1)}+1)}{8b^5}\end{matrix}\\ \dfrac{iq_{21}^{(1)}}{2b}; & 0\end{matrix}\right\|\begin{matrix}t^{-2}+\\+O(t^{-3})\end{matrix}\right]\times$$
$$\times\left\|\begin{matrix}t; & -i\dfrac{q_{12}^{(1)}}{2b}-\dfrac{\lambda_{12}^{(1)}}{t}\\ 0; & 1\end{matrix}\right\|\exp(at). \tag{4.6}$$

§ 5. Let us consider the equation

$$\frac{dX(t)}{dt}=X(t)\left\{\left\|\begin{matrix}a & 0\\ 1 & a\end{matrix}\right\|+\sum_{k=1}^{\infty}H^{(k)}t^{-k}\right\}. \tag{5.1}$$

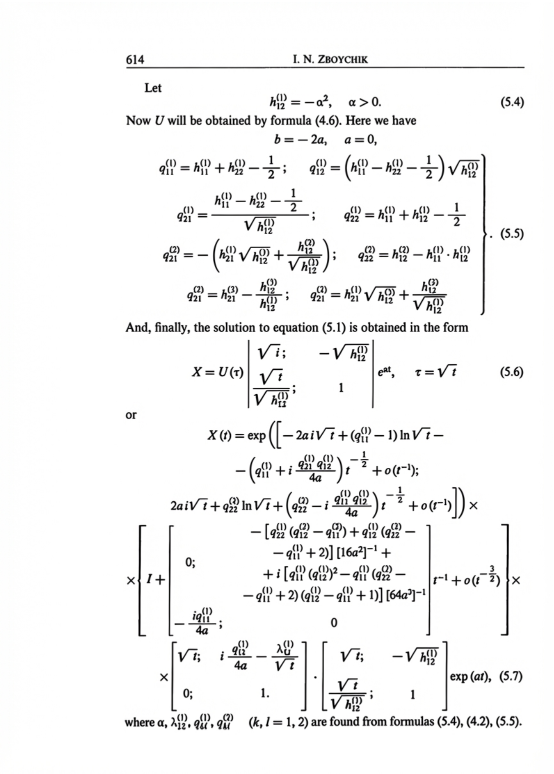
Following Khoroshilov, we introduce a new independent variable $\tau=\sqrt{t}$ and a new unknown matrix $U(\tau)$ using the formula

$$X(\tau)=U(\tau)\left\|\begin{matrix}1; & -\sqrt{h_{12}^{(1)}}\\ \dfrac{1}{\sqrt{h_{12}^{(1)}}}; & 1\end{matrix}\right\|\cdot\left\|\begin{matrix}\tau & 0\\ 0 & 1\end{matrix}\right\|\cdot e^{a\tau^2}. \tag{5.2}$$

Then we obtain

$$\frac{dU}{d\tau}=U\left\{\left\|\begin{matrix}-2\sqrt{h_{12}^{(1)}}; & 0\\ 0; & 2\sqrt{h_{12}^{(1)}}\end{matrix}\right\|+\left\|\begin{matrix}h_{11}^{(1)}+h_{12}^{(1)}-\dfrac{1}{2}; & \left(h_{11}^{(1)}-h_{22}^{(1)}-\dfrac{1}{2}\right)\sqrt{h_{12}^{(1)}}\\ \dfrac{h_{11}^{(1)}-h_{22}^{(1)}-\dfrac{1}{2}}{\sqrt{h_{12}^{(1)}}}; & h_{11}^{(1)}+h_{22}^{(1)}-\dfrac{1}{2}\end{matrix}\right\|\tau^{-1}+\right.$$
$$\left.+\left\|\begin{matrix}-\left(h_{21}^{(1)}\sqrt{h_{12}^{(1)}}+\dfrac{h_{12}^{(2)}}{\sqrt{h_{12}^{(1)}}}\right); & h_{12}^{(2)}-h_{21}^{(1)}h_{12}^{(1)}\\ h_{21}^{(1)}-\dfrac{h_{13}^{(2)}}{h_{12}^{(1)}}; & h_{21}^{(1)}\sqrt{h_{12}^{(1)}}+\dfrac{h_{12}^{(2)}}{\sqrt{h_{12}^{(1)}}}\end{matrix}\right\|\tau^{-2}+O(\tau^{-3})\right\}. \tag{5.3}$$

Figure 10: Figure 10



Let

$$h_{12}^{(1)} = -\alpha^2, \quad \alpha > 0. \tag{5.4}$$

Now U will be obtained by formula (4.6). Here we have

$$b = -2a, \quad a = 0,$$

$$\left.\begin{aligned}
&q_{11}^{(1)} = h_{11}^{(1)} + h_{22}^{(1)} - \frac{1}{2}; \qquad q_{12}^{(1)} = \left(h_{11}^{(1)} - h_{22}^{(1)} - \frac{1}{2}\right)\sqrt{h_{12}^{(1)}}\\
&q_{21}^{(1)} = \frac{h_{11}^{(1)} - h_{22}^{(1)} - \frac{1}{2}}{\sqrt{h_{12}^{(1)}}}; \qquad q_{22}^{(1)} = h_{11}^{(1)} + h_{12}^{(1)} - \frac{1}{2}\\
&q_{21}^{(2)} = -\left(h_{21}^{(1)}\sqrt{h_{12}^{(1)}} + \frac{h_{12}^{(2)}}{\sqrt{h_{12}^{(1)}}}\right); \qquad q_{22}^{(2)} = h_{12}^{(2)} - h_{11}^{(1)} \cdot h_{12}^{(1)}\\
&q_{21}^{(2)} = h_{21}^{(3)} - \frac{h_{12}^{(3)}}{h_{12}^{(1)}}; \qquad q_{21}^{(2)} = h_{21}^{(1)}\sqrt{h_{12}^{(1)}} + \frac{h_{12}^{(3)}}{\sqrt{h_{12}^{(1)}}}
\end{aligned}\right\}. \tag{5.5}$$

And, finally, the solution to equation (5.1) is obtained in the form

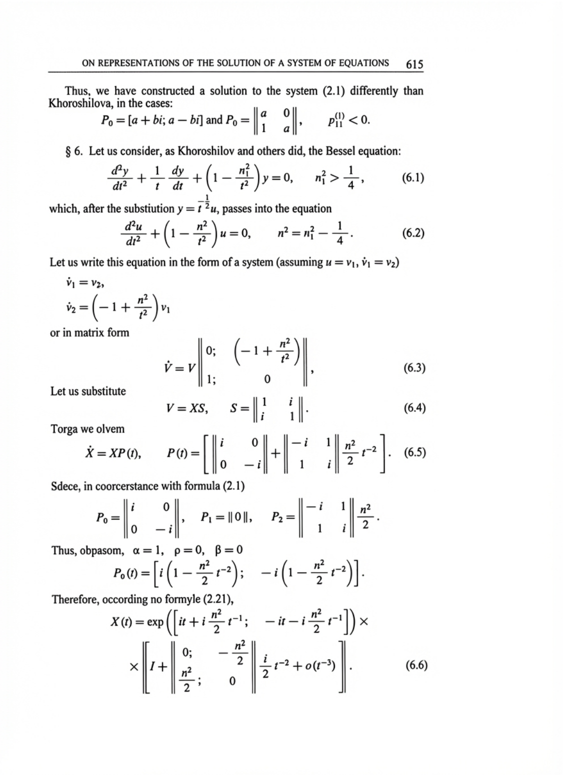
$$X = U(\tau)\begin{vmatrix} \sqrt{i}; & -\sqrt{h_{12}^{(1)}} \\ \frac{\sqrt{t}}{\sqrt{h_{12}^{(1)}}}; & 1 \end{vmatrix} e^{at}, \quad \tau = \sqrt{t} \tag{5.6}$$

or

$$\begin{aligned}
X(t) = \exp\Bigg(\Bigg[&-2ai\sqrt{t} + (q_{11}^{(1)} - 1)\ln\sqrt{t} - \\
&-\left(q_{11}^{(1)} + i\,\frac{q_{21}^{(1)}q_{12}^{(1)}}{4a}\right)t^{-\frac{1}{2}} + o(t^{-1}); \\
&2ai\sqrt{t} + q_{22}^{(2)}\ln\sqrt{t} + \left(q_{22}^{(2)} - i\,\frac{q_{11}^{(1)}q_{12}^{(1)}}{4a}\right)t^{-\frac{1}{2}} + o(t^{-1})\Bigg]\Bigg) \times \\
\times\left\{ I + \begin{bmatrix} 0; & \begin{matrix} -\left[q_{22}^{(1)}(q_{12}^{(2)} - q_{11}^{(3)}) + q_{12}^{(1)}(q_{22}^{(2)} - \right. \\ \left. - q_{11}^{(1)} + 2)\right][16a^2]^{-1} + \\ + i\left[q_{11}^{(1)}(q_{12}^{(1)})^2 - q_{11}^{(1)}(q_{22}^{(2)} - \right. \\ \left. - q_{11}^{(1)} + 2)(q_{12}^{(1)} - q_{11}^{(1)} + 1)\right][64a^3]^{-1} \end{matrix} \\ -\frac{iq_{11}^{(1)}}{4a}; & 0 \end{bmatrix} t^{-1} + o(t^{-\frac{3}{2}}) \right\} \times \\
\times\begin{bmatrix} \sqrt{t}; & i\,\frac{q_{12}^{(1)}}{4a} - \frac{\lambda_{U}^{(1)}}{\sqrt{t}} \\ 0; & 1. \end{bmatrix} \cdot \begin{bmatrix} \sqrt{t}; & -\sqrt{h_{12}^{(1)}} \\ \frac{\sqrt{t}}{\sqrt{h_{12}^{(1)}}}; & 1 \end{bmatrix} \exp(at), \qquad (5.7)
\end{aligned}$$

where α, $\lambda_{12}^{(1)}$, $q_{kl}^{(1)}$, $q_{kl}^{(2)}$ $(k, l = 1, 2)$ are found from formulas (5.4), (4.2), (5.5).

Figure 11: Figure 11



Thus, we have constructed a solution to the system (2.1) differently than Khoroshilova, in the cases:

$$P_0 = [a + bi;\, a - bi] \text{ and } P_0 = \begin{Vmatrix} a & 0 \\ 1 & a \end{Vmatrix}, \qquad p_{11}^{(1)} < 0.$$

§ 6. Let us consider, as Khoroshilov and others did, the Bessel equation:

$$\frac{d^2y}{dt^2} + \frac{1}{t}\,\frac{dy}{dt} + \left(1 - \frac{n_1^2}{t^2}\right) y = 0, \qquad n_1^2 > \frac{1}{4}, \tag{6.1}$$

which, after the substiution $y = t^{-\frac{1}{2}}u$, passes into the equation

$$\frac{d^2u}{dt^2} + \left(1 - \frac{n^2}{t^2}\right) u = 0, \qquad n^2 = n_1^2 - \frac{1}{4}. \tag{6.2}$$

Let us write this equation in the form of a system (assuming $u = v_1$, $\dot{v}_1 = v_2$)

$$\dot{v}_1 = v_2,$$

$$\dot{v}_2 = \left(-1 + \frac{n^2}{t^2}\right) v_1$$

or in matrix form

$$\dot{V} = V \begin{Vmatrix} 0; & \left(-1 + \frac{n^2}{t^2}\right) \\ 1; & 0 \end{Vmatrix}, \tag{6.3}$$

Let us substitute

$$V = XS, \qquad S = \begin{Vmatrix} 1 & i \\ i & 1 \end{Vmatrix}. \tag{6.4}$$

Torga we olvem

$$\dot{X} = XP(t), \qquad P(t) = \left[\begin{Vmatrix} i & 0 \\ 0 & -i \end{Vmatrix} + \begin{Vmatrix} -i & 1 \\ 1 & i \end{Vmatrix} \frac{n^2}{2} t^{-2} \right]. \tag{6.5}$$

Sdece, in coorcerstance with formula (2.1)

$$P_0 = \begin{Vmatrix} i & 0 \\ 0 & -i \end{Vmatrix}, \quad P_1 = \|0\|, \quad P_2 = \begin{Vmatrix} -i & 1 \\ 1 & i \end{Vmatrix} \frac{n^2}{2}.$$

Thus, obpasom, $\alpha = 1, \quad \rho = 0, \quad \beta = 0$

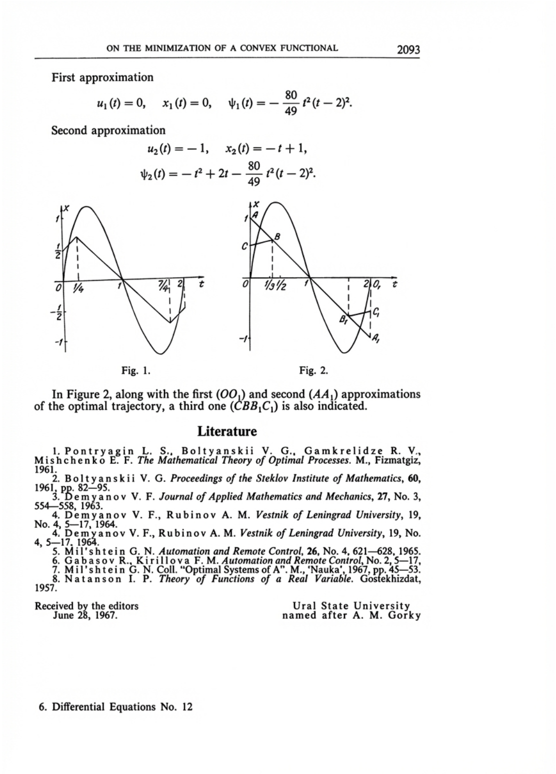
$$P_0(t) = \left[i\left(1 - \frac{n^2}{2} t^{-2}\right); \quad -i\left(1 - \frac{n^2}{2} t^{-2}\right) \right].$$

Therefore, according no formyle (2.21),

$$X(t) = \exp\left(\left[it + i\,\frac{n^2}{2} t^{-1}; \quad -it - i\,\frac{n^2}{2} t^{-1} \right]\right) \times$$

$$\times \left\| \left[I + \begin{Vmatrix} 0; & -\frac{n^2}{2} \\ \frac{n^2}{2}; & 0 \end{Vmatrix} \frac{i}{2} t^{-2} + o(t^{-3}) \right] \right\|. \tag{6.6}$$

Figure 12: Figure 12



First approximation

$$u_1(t) = 0, \quad x_1(t) = 0, \quad \psi_1(t) = -\frac{80}{49} t^2 (t-2)^2.$$

Second approximation

$$u_2(t) = -1, \quad x_2(t) = -t+1,$$

$$\psi_2(t) = -t^2 + 2t - \frac{80}{49} t^2 (t-2)^2.$$

Fig. 1.

Fig. 2.

In Figure 2, along with the first (OO_1) and second (AA_1) approximations of the optimal trajectory, a third one (CBB_1C_1) is also indicated.

Literature

1. Pontryagin L. S., Boltyanskii V. G., Gamkrelidze R. V., Mishchenko E. F. *The Mathematical Theory of Optimal Processes*. M., Fizmatgiz, 1961.
2. Boltyanskii V. G. *Proceedings of the Steklov Institute of Mathematics*, **60**, 1961, pp. 82—95.
3. Demyanov V. F. *Journal of Applied Mathematics and Mechanics*, **27**, No. 3, 554—558, 1963.
4. Demyanov V. F., Rubinov A. M. *Vestnik of Leningrad University*, 19, No. 4, 5—17, 1964.
4. Demyanov V. F., Rubinov A. M. *Vestnik of Leningrad University*, 19, No. 4, 5—17, 1964.
5. Mil'shtein G. N. *Automation and Remote Control*, **26**, No. 4, 621—628, 1965.
6. Gabasov R., Kirillova F. M. *Automation and Remote Control*, No. 2, 5—17,
7. Mil'shtein G. N. Coll. "Optimal Systems of A". M., 'Nauka', 1967, pp. 45—53.
8. Natanson I. P. *Theory of Functions of a Real Variable*. Gostekhizdat, 1957.

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Figure 13: Figure 13