NONLOCAL SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

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Abstract

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NONLOCAL SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

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Consider, in a bounded domain Ω of Euclidean space R^n, formally self-adjoint elliptic operators of order $2m$

$$A(t,x,D)u(x)=\sum_{|\alpha|,|\beta|\le m} D^\alpha\big(a_{\alpha,\beta}(t,x)D^\beta u(x)\big)\qquad (t\in[0,T]) \tag{1}$$

with homogeneous boundary conditions

$$B_j(x,D)u(x)\big|_\Gamma \equiv \sum_{|\alpha|\le k_j} b^j_\alpha(x)D^\alpha u(x)\big|_\Gamma=0\qquad (j=1,\dots,m), \tag{2}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$; $|\alpha|=\alpha_1+\dots+\alpha_n$; $D^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$; the functions $a_{\alpha,\beta}(t,x)$ and $b^j_\alpha(x)$ are real; $a_{\alpha,\beta}(t,x)=(-1)^{|\alpha|+|\beta|}\cdot a_{\beta,\alpha}(t,x)$; $k_j\le 2m-1$, and

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}(t,x)\xi^\alpha\xi^\beta \ge a\sum_{i=1}^n \xi_i^{2m},\qquad a=\text{const}>0,$$

for every real $\xi=(\xi_1,\dots,\xi_n)$; $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$. Let the boundary Γ of the domain Ω be continuously differentiable $2m$ times. Let the boundary conditions (2) cover the differential expression (1)[1] (the Shapiro-Lopatinskii condition).

Let the elliptic operator $A(t)$[1,2], generated by the system $\{A(t,x,D),\ B_j(x,D)\}$, be a self-adjoint operator in $L_2(\Omega)$. (This is a condition on the boundary operators B_j. For example, if $B_j(x,D)=\partial^j/\partial n^j$ and $j=0,\dots,m-1$ (the Dirichlet problem) or $j=m,\dots,2m-1$ (the Neumann problem), then this condition is satisfied.) As is known[2], the domain of definition of the elliptic operator $A(t)$ is a closed subspace of $W_2^{2m}(\Omega)$.

By $C(t, x, D)$ we denote any differential operator of order not exceeding m:

$$C(t, x, D) = \sum_{|\alpha| \leq m} C_\alpha(t, x) D^\alpha u(x).$$

In the article the following problems are investigated:

I. For $n < 2m$, find a function $u(t, x)$, defined in the cylinder $Q[0, T] \times \Omega$, satisfying the equation

$$\frac{\partial^2 u(t, x)}{\partial t^2} + A(t, x, D)u(t, x) + C(t, x, D)u(t, x) =$$

$$= F(t, x, u, \dots, D^\delta u, |u|^2, \dots, |D^\delta u|^2) \frac{\partial u(t, x)}{\partial t} + \frac{d\Phi(|u|^2)}{dr} u + \tilde{f}(t, x), \qquad (3)$$

the initial conditions

$$u(t, x)\big|_{t=0} = u_0(x), \qquad \partial u(t, x)/\partial t\big|_{t=0} = u_1(x) \qquad (4)$$

and the boundary conditions (2).

II. For $2m \leq n < 4m$, find a function $u(t, x)$ satisfying the equation

$$\frac{\partial^2 u(t, x)}{\partial t^2} + A(t, x, D)u(t, x) + C(t, x, D)u(t, x) = d\Phi\left(|\nu(t, x)|^2\right)/d\tau\, u(t, x) + f(t, x) \qquad (5)$$

and the conditions (2), (4).

Problems I and II reduce to the abstract Cauchy problem

$$\frac{d^2 u(t)}{dt^2} + A(t)u(t) = f(t, u(t), du(t)/dt), \qquad u(0) = u_0, \qquad u'(0) = u_1 \qquad (6)$$

in the Hilbert space $L_2(\Omega)$. By $L_2(\Omega)$ we denote the space of complex-valued functions summable with the square of the modulus. Problem (6) in a Hilbert space H was investigated in the author' s paper ([3]). Particular cases of problem (6) were investigated in papers ([4–6]). (The results obtained in ([3]), even for these particular cases, turned out to be new.) In paper ([7]) problem (6) was investigated under other assumptions. Problem (6) was also investigated in a Banach space in the author' s papers ([8],[9]). We state one theorem from ([3]), by means of which the solvability of problems I and II as a whole is established.

Theorem 1. *Suppose that the following conditions are satisfied:*

1°. *For each $t \in [0,T]$ the operator $A(t)$ is self-adjoint and positive definite in H, and for all $t \in [0,T]$ and $u \in D(A(t))$ the inequality*

$$(A(t)u, u) \geq \gamma(u, u) \qquad (\gamma = \text{const} > 0).$$

holds.

2°. *The operator $A(t)$ has a domain of definition $D(A)$ independent of t; the operator-function $A(t)A^{-1}(0)$ is strongly continuously differentiable.*

3°. *The operator $f(t,u,v)$ maps $[0,T] \times H(A) \times H(A^{1/2})$ into H and is bounded; for every function $u(t)$ continuous in $H(A)$, continuously differentiable in $H(A^{1/2})$, and every function $v(t)$ continuous in $H(A^{1/2})$, continuously differentiable in H, the vector-function $f(t,u(t),v(t))$ is continuously differentiable in H; for any two pairs of functions $u_1(t), u_2(t), v_1(t), v_2(t)$, respectively satisfying these smoothness conditions, the inequality*

$$\left\|\frac{d}{dt}\left[f(t,u_1(t),v_1(t)) - f(t,u_2(t),v_2(t))\right]\right\| \leq$$

$$\leq C(R)[\|A(0)(u_1(t)-u_2(t))\| + \|A^{1/2}(0)(u_1'(t)-u_2'(t))\| +$$

$$+ \|A^{1/2}(0)(v_1(t)-v_2(t))\| + \|v_1'(t)-v_2'(t)\|],$$

as soon as

$$\|A(0)u_i(t)\| + \|A^{1/2}(0)u_i'(t)\| \leq R \quad \text{and} \quad \|A^{1/2}(0)v_i(t)\| + \|v_i'(t)\| \leq R \quad (i=1,2).$$

4°. *For any functions $u(t)$ continuous in $H(A)$, continuously differentiable in $H(A^{1/2})$, twice continuously differentiable in H, one has*

$$\operatorname{Re}\int_0^t \left(f\left(\tau, u(\tau), \frac{du(\tau)}{d\tau}\right), \frac{du(\tau)}{d\tau}\right) d\tau \leq C\left[1 + \int_0^t \left(\|A^{1/2}(0)u(\tau)\|^2 + \left\|\frac{du(\tau)}{d\tau}\right\|^2\right) d\tau\right];$$

for all functions $u(t)$ satisfying these smoothness conditions, the inequality

$$\left\|\frac{d}{dt}f(t,u(t),u'(t))\right\| \leq C(R)\left[1 + \|A(0)u(t)\| + \left\|A^{1/2}(0)\frac{du(t)}{dt}\right\| + \left\|\frac{d^2u(t)}{dt^2}\right\|\right], \tag{7}$$

holds as soon as

$$\|A^{1/2}(0)u(t)\| + \|du(t)/dt\| \leq R.$$

5°. $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$.

Then problem (6) has a unique solution on $[0, T]$.

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* By $H(A^\alpha)$ is denoted the Hilbert space consisting of elements of $D(A^\alpha)$ with scalar product $(u, v)_{A^\alpha} = (A^\alpha(0)u, A^\alpha(0)v)$.

This theorem makes it possible to establish the following theorems.

Theorem 2. *Let the following conditions be satisfied:*

1°. $a_{\alpha,\beta}(t, x) \in C^{(\alpha)}(\overline{\Omega})$, $b^i_\alpha \in C^{2m-k_j}$; the functions $D^l a_{\alpha,\beta}(t, x)$ ($|l| \leq |\alpha|$), $C_\alpha(t, x)$, and their first derivatives with respect to t are continuous in $\overline{Q}$.

2°. $|\delta| \leq (2m - n)/4$; the function $F(t, x, z_s, r_s)$ is continuous together with its derivatives with respect to t, z_s, r_s in the domain $\{t \in [0, T],\ x \in \overline{\Omega},\ |z_s| < \infty,\ 0 \leq r_s < \infty\}$; the function $\Phi(r)$ is twice continuously differentiable on $[0, \infty)$; the function $f(t, x)$ is continuous together with its derivative with respect to t in the domain $\{t \in [0, T],\ x \in \overline{\Omega}\}$; the functions $\partial F/\partial t$, $\partial F/\partial z_s$, $\partial F/\partial r_s$, and $d^2\Phi/dr^2$ satisfy, in z_s, r_s, the Lipschitz condition in every bounded set $|z_s| \leq R$, $0 \leq r_s \leq R$, with Lipschitz constants depending only on R.

3°. $\operatorname{Re} F(t, x, z_s, r_s) \leq C$, $\operatorname{Re} \Phi(r) \leq C(1 + r)$.

4°. $u_0(x) \in W^{2m}_2(\Omega, B)$, $u_1(x) \in W^m_2(\Omega, B)$.

Then problem I has a unique solution in the cylinder $Q = [0, T] \times \Omega$.

Theorem 3. *Let conditions* 1° *and* 4° *of Theorem 2 be satisfied. Suppose further:*

5°. The function $f(t, x)$, together with its derivative $\partial f(t, x)/\partial t$, is continuous in $\overline{Q}$; the function $\Phi(r)$ is three times continuously differentiable on $[0, \infty)$; the following estimates hold:*

$$\operatorname{Re} \Phi(r) \leq C(1 + r),$$

$$|\Phi'(r)| \leq C(1 + r^q), \qquad |\Phi''(r)| \leq C(1 + r^{q-1}), \qquad |\Phi'''(r)| \leq C(1 + r^{q-2}),$$

where $q = 2m(n - 2m)^{-1}$ for $n > 2m$; q is arbitrary for $n = 2m$.

Then problem II has a unique solution in the cylinder $Q = [0, T] \times \Omega$.

Theorems 2 and 3 improve and generalize the results of the papers ([4,5]). For example, in ([4]), for $n > 2m$, it is assumed that $q = m(n - 2m)^{-1}$. Similar results were obtained in the article ([10]).

We indicate the plan of the proof.

Consider the nonlinear operator

$$f(t,u,v) = -C(t,x,D)u(x) + \lambda u(x) +$$

$$+F(t,x,u,\dots,D^{\delta}u,|u|^2,\dots,|D^{\delta}u|^2)v + \frac{d\Phi(|u|^2)}{dr}u + f(t,x)$$

in the space $L_2 \times L_2$. For lack of space, we study the "principal" term $F(u) = \Phi'(|u|^2)u$ of the operator $f(t,u,v)$. Since the main restrictions and difficulties arise in establishing the weakened two-sided estimate (7), for the operator $F(u)$, when $n > 2m$, we verify inequality (7). From

$$\frac{d}{dt}F(u(t)) = \Phi'(|u(t)|^2)u'(t) + \Phi''(|u(t)|^2)[|u(t)|^2u'(t) + u^2(t)(\overline{u(t)})']$$

it follows that

$$\left\|\frac{d}{dt}F(u(t))\right\|_2^2 \le C\int_{\Omega}|u(t,x)|^{4q}\left|\frac{\partial u(t,x)}{\partial t}\right|^2 dx.$$

Since $u(t,x) \in W_2^{2m}$, $\partial u(t,x)/\partial t \in W_2^m$, and $\|u(t)\|_{W_2^{2m}} + \|\partial u(t)/\partial t\|_2 \le R$, then for $2m < n < 4m$, applying the results of the article ([11]), we obtain the inequality

$$\left\|\frac{d}{dt}F(u(t))\right\|_2^2 \le C\|u(t)\|_C^{4q}\left\|\frac{du(t)}{dt}\right\|_2^2 \le C(R)\|u(t)\|_C^{4q} \le$$

$$\le C(R)\big(\|u(t)\|_{W_p^l}^{\tau}\|u(t)\|_p^{1-\tau}\big)^{4q} \le C(R)\|u(t)\|_{W_p^l}^{4q\tau},$$

where $p = 2n/(n-2m)$, $l > (n-2m)/2$, $\tau = (n-2m)/2l$. On the other hand, by virtue of ([12,13]), for any $0 \le l \le 2m$, the inequality

$$\|u\|_{W_p^l} \le C\|u\|_{W_2^{2m}}^{l/2m}\|u\|_p^{(2m-l)/2m}.$$

Thus, for any $l \in ((n-2m)/2, 2m]$ we have

$$\left\|\frac{d}{dt}F(u(t))\right\|_2^2 \le C(R)\|u(t)\|_{W_2^{2m}}^{\frac{l}{2m}4q\tau} \le C(R)\|u(t)\|_{W_2^{2m}}^2,$$

whence, in turn, inequality (7) follows.

Remark. The results obtained carry over to quasilinear hyperbolic systems with a strongly elliptic principal part.

Let us note that, for the equation

$$\partial^2 u/\partial t^2 - a\Delta u + b(t)u^p = 0, \qquad \Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2, \quad b(t) \geq 0, \tag{9}$$

which occurs in quantum field theory, the results of papers (4–7, 10) are applicable only for $p = 1$ and $p = 3$. As is easy to see, Theorem 3 is applicable to equation (9) also for $p = 5$.

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Note: Figure translations are in progress. See original paper for figures.