A GENERALIZATION OF THE SHEFFER-DUFFIN PROBLEM TO FINITE FUNCTIONALS

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MATHEMATICS

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A GENERALIZATION OF THE SHEFFER-DUFFIN PROBLEM TO FINITE FUNCTIONALS

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In the work (^{1}), Sheffer and Duffin published the following result, which they regarded as a strengthening of a well-known theorem of V. A. Markov (^{2}): let $\{\lambda_i\}_{i=0}^n$ be the points of extremum of the polynomial $L_n(x) = \cos n \arccos x$ on the interval $[-1, 1]$; then, among polynomials of degree n satisfying the condition $|P_n(\lambda_i)| \leq 1$ $(i = 0, 1, \dots, n)$, the maximum of the k-th derivative $(k = 1, 2, \dots, n)$ on the interval $[-1, 1]$ is attained on the polynomial $L_n(x)$, i.e.

$$|P_n^{(k)}(x)| \leq L_n^{(k)}(1).$$

In the present article the general problem formulated below is solved and some of its special cases are considered. In what follows the consideration is carried out on the interval $[0, 1]$.

I. **Problem.** Let on the set of algebraic polynomials with real coefficients of degree not exceeding n there be given a finite functional $F = (\mu_i)_0^n$, where $\mu_i = F(x^i)$, (μ_i) are real numbers, and let on $[0, 1]$ be given $n + 1$ consecutive points $0 \leq t_0 < t_1 < \dots < t_n \leq 1$, and to each point there be assigned a nonnegative number y_i. In the class $\mathcal{P}_t$ of polynomials $P_n(x)$ of degree not exceeding n satisfying the condition $|P_n(t_i)| \leq y_i$ $(i = 0, 1, \dots, n)$, find that polynomial on which $\sup F[P_n] = N_F$ is attained.

Any polynomial solving the problem will be called extremal.

Any polynomial $P_n(x) = \sum_{j=0}^n a_j^{(P)} x^j \in \mathcal{P}_t$ can be represented in the form

$$P_n(x) = \sum_{i=0}^n \frac{\delta_i}{R'_{n+1}(t_i)} R_{n+1,i}(x), \tag{1}$$

where $|\delta_i| \leq y_i$;

$$R_{n+1}(x) = \prod_{m=0}^n (x - t_m); \qquad R_{n+1,i}(x) = \frac{R_{n+1}(x)}{x - t_i}; \qquad F[P_n] = \sum_{i=0}^n \delta_i K_i;$$

$$K_i = \frac{F[R_{n+1,i}]}{R'_{n+1}(t_i)}.$$

Put

$$M_n(x) = \sum_{i=0}^{n} \frac{y_i}{|R'_{n+1}(t_i)|} R_{n+1,i}(x) = \sum_{i=0}^{n} a_i^{(M)} x^i \in \mathcal{P}_t.$$

It is easy to see that for every polynomial $P_n(x) \in \mathcal{P}_t$ the inequalities

$$|a_j^{(P)}| \le |a_j^{(M)}|, \qquad j = 0, 1, \dots, n. \tag{2}$$

hold.

From consideration of formula (1) it follows easily that

Theorem 1. *For every functional F, an extremal polynomial is any polynomial of the form (1), where $\delta_i = y_i \operatorname{sign} K_i$ for $K_i \ne 0$; δ_i arbitrary*

from $[-y_i, y_i]$, if $K_i = 0$.

$$N_F = \sum_{i=0}^{n} y_i |K_i|.$$

Remark. If all $K_i \ne 0$, then the extremal polynomial is unique; if some $K_i = 0$, then below in the extremal polynomial we shall put $\delta_i = y_i$. In this way we ensure uniqueness of the extremal polynomial in all cases, and the number of extremal polynomials for all possible functionals does not exceed 2^{n+1}.

Let $\mu_i = \mu_i(\xi)$ be a continuous function of the real argument ξ, $-\infty < \xi < +\infty$, $i = 0, 1, \dots, n$. Then $F_\xi[P_n]$ is a continuous function of ξ, and in this case, according to Theorem 1, the problem of finding the extremal polynomials $\{G_n(x, \xi)\}$ is practically equivalent to the problem of finding the points of sign change of the functions $\{F_\xi[R_{n+1,i}]\}_{i=0}^{n}$.

We proceed to the consideration of concrete interval-functionals.

II. $F_\xi[P_n] = P_n^{(k)}(\xi)$, i.e.

$$\mu_i = 0, \quad i = 0, 1, \dots, k-1; \qquad \mu_i = \frac{i!}{(i-k)!} \xi^{i-k}, \quad i = k, k+1, \dots, n. \tag{3}$$

Denote by $\{x_i^{(l,k)}\}_{i=1}^{n-k}$ the roots of the polynomial $R_{n+1,l}^{(k)}(x)$ $(l = 0, 1, \dots, n)$.

Theorem 2. For $\xi \in (-\infty, x_1^{(n,k)})$, $(x_{n-k}^{(0,k)}, +\infty)$, $(x_i^{(0,k)}, x_{i+1}^{(n,k)})$ $(i = 1, 2, \dots, n-k-1)$, one of the polynomials $\pm M_n(x)$ is extremal; for $\xi \in (x_i^{(l,k)}, x_i^{(l-1,k)})$ $(i = 1, 2, \dots, n-k;\ l = 1, 2, \dots, n)$ the polynomial

$$\Lambda_{l,n}(x) = M_n(x) + 2\sum_{m=l}^{n} \frac{(-1)^{n-m+1} y_m}{R'_{n+1}(t_m)} R_{n+1,m}(x);$$

is extremal up to sign; for $\xi \in \{x_i^{(l,k)}\}_{i=1}^{n-k}$ $(l = 0, 1, \dots, n)$ one of the polynomials $\pm M_n(x)$ or a polynomial of the form $\Lambda_{l,n}(x)$ is extremal (chosen in accordance with the remark).

The proof of the theorem follows from the interlacing of the roots of the polynomials $\{R_{n+1,l}^{(k)}(x)\}_{l=0}^{n}$.

Corollary 1. For $k = 1, 2, \dots, n-1$, the number of extremal polynomials (up to sign) is not greater than $n+1$; for $k = n$, on the whole real axis the extremal polynomial is $M'_n(x)$.

Corollary 2. If all $y_i > 0$, then each of the extremal polynomials $\Lambda_{l,n}(x)$ has n points (among $\{t_i\}_{i=0}^{n}$) at which the values $|\Lambda_{l,n}(t_i)| = y_i$ are attained with successively opposite signs.

Corollary 3. The sum of the lengths of the intervals located on $[0, 1]$, at each point of which one of the polynomials $\pm M_n(x)$ is extremal, is equal to $1 - (1 - k/n)(t_n - t_0)$; the sum of the lengths of the intervals at whose points the polynomial $\Lambda_{l,n}(x)$ $(l = 1, 2, \dots, n)$ is extremal up to sign is equal to $(1 - k/n)(t_l - t_{l-1})$.

Corollary 4. For any $\xi > 0$

$$\sup_{P_n \in \mathcal{P}_t^y} F_\xi[P_n] = N_F(\xi) < |M_n^{(k)}(-\xi)| = N_F(-\xi).$$

III. If the points $\{t_i\}_{i=0}^{n}$ are taken to be the points of extremum of the polynomial

$$T_n(x) = \cos n \arccos(2x - 1)$$

(denote them by $\{\tau_i\}_{i=0}^{n}$), and $\{y_i\}_{i=0}^{n} = 1$, then Theorems 1, 2 and their corollaries contain assertions directly supplementing the result of Sheffer and Duffin (1). In this case the extremal polynomials have the form

$$P_{l,n}(x) = T_n(x) + \frac{x(x-1)T'_n(x)}{n^2 2^{2n-2}} \sum_{m=l}^{n} \frac{(-1)^{n-m+1}}{R'_{n+1}(\tau_m)(x - \tau_m)}, \tag{4}$$

$$l = 0, 1, \dots, n \qquad (P_{0,n} = -T_n(x)).$$

Besides the properties possessed by all extremal polynomials of the functional (3) (see Corollaries 2 and 3 of Theorem 2), the polynomials $\{P_{l,n}(x)\}$ have certain specific properties.

Theorem 3. $P'_{l,n}(\tau_i)P'_{l,n}(\tau_{i+1}) < 0$ $(i = 1, 2, \dots, n-2;\ l = 1, 2, \dots, n)$.

Proof. It suffices to consider $l > n/2$, since

$$P_{l,n}(x) = (-1)^{n-1}P_{n-l+1,n}(1-x) \qquad (l = 1, 2, \dots, n).$$

We shall show that, for $n/2 < l \le n$,

$$\operatorname{sign} P'_{l,n}(\tau_m) = (-1)^{n-m} \qquad (m = 1, 2, \dots, n-1).$$

Taking into account that

$$R'_{n+1}(\tau_m) = (-1)^{n-m}\frac{n}{2^{2n-1}} \qquad (m = 1, 2, \dots, n-1);$$

$$R'_{n+1}(0) = (-1)^n\frac{2n}{2^{2n-1}}; \qquad R'_{n+1}(1) = \frac{2n}{2^{2n-1}},$$

we have (see (4))

$$P_{l,n}(x) = T_n(x) - \frac{2^{2n-1}R_{n+1}(x)}{n}\left[2\sum_{m=l}^{n}\frac{1}{x-\tau_m} - \frac{1}{x-1}\right].$$

Let $\tau_i < \tau_l$ $(i = 1, 2, \dots, l-1)$. Then

$$P'_{l,n}(\tau_i) = (-1)^{n-i-1}\left[2\sum_{m=l}^{n}\frac{1}{\tau_i-\tau_m} - \frac{1}{\tau_i-1}\right]$$

and

$$\operatorname{sign} P'_{l,n}(\tau_i) = (-1)^{n-i} \qquad (i = 1, 2, \dots, l-1).$$

Let $\tau_i \ge \tau_l$ $(i = l, l+1, \dots, n-1)$. Note that

$$\left(\frac{R_{n+1}(x)}{x-\tau_i}\right)'_{x=\tau_i} = \frac{1-2\tau_i}{4\tau_i(1-\tau_i)}R'_{n+1}(\tau_i);$$

$$P'_{l,n}(\tau_i) = (-1)^{n-i-1}\left[2\sum_{\substack{m=l\\ m\ne i}}^{n}\frac{1}{\tau_i-\tau_m} - \frac{1}{\tau_i-1} + \frac{1-2\tau_i}{2\tau_i(1-\tau_i)}\right],$$

and hence $\operatorname{sign} P'_{l,n}(\tau_l) = (-1)^{n-l}$. Put $\tau_i > \tau_l$ $(i = l+1, \dots, n-1)$. Introduce the notation

$$\Pi_i(x) = \prod_{\substack{m=l\\ m\ne i}}^{n}(x-\tau_m); \qquad \Lambda_l(x) = \prod_{m=0}^{l-1}(x-\tau_m).$$

Then

$$\Pi_i(x)\Lambda_l(x)(x-\tau_i) = R_{n+1}(x); \qquad \sum_{\substack{m=l\\ m\neq i}}^{n} \frac{1}{x-\tau_m} = \frac{\Pi_i'(x)}{\Pi_i(x)};$$

$$\Pi_i(\tau_i) = \frac{R_{n+1}'(\tau_i)}{\Lambda_l(\tau_i)}; \qquad \Pi_i'(\tau_i) = \frac{R_{n+1}'(\tau_i)}{\Lambda_l(\tau_i)}\left[\frac{1-2\tau_i}{4\tau_i(1-\tau_i)} - \frac{\Lambda_l'(\tau_i)}{\Lambda_l(\tau_i)}\right];$$

$$P_{l,n}'(\tau_i) = (-1)^{n-i}\left[\frac{2\Lambda_l'(\tau_i)}{\Lambda_l(\tau_i)} - \frac{1}{\tau_i}\right].$$

Noting that

$$\frac{2\Lambda_l'(x)}{\Lambda_l(x)} - \frac{1}{x} > 0 \quad \text{for } x \geq \tau_{l-1},$$

we obtain

$$\operatorname{sign} P_{l,n}'(\tau_i) = (-1)^{n-i} \qquad (i = l+1, l+2, \dots, n-1),$$

and the theorem is proved.

Corollary 1. The polynomials $P_{l,n}'(x)$ $(l = 1, 2, \dots, n)$ have one root in each of the segments $[\tau_i, \tau_{i+1}]$ $(i = 1, 2, \dots, n-2)$.

Corollary 2. For $n/2 < l < n$

$$\sum_{m=1}^{n-1} |P_{l,n}'(\tau_m)| = \frac{1}{2}\left(|P_{l,n}'(0)| + |P_{l,n}'(1)|\right).$$

Corollary 3.

$$\sum_{m=1}^{n-1} |P_{n,n}'(\tau_m)| = \frac{1}{2}\left(|P_{n,n}'(0)| - P_{n,n}'(1)\right) = \frac{2}{3}(n^2-1).$$

Remark. With the aid of Theorems 1-3 the proof of the Schaeffer-Duffin theorem is simplified and can be carried out by purely real-variable means (by analogy with the proof of S. N. Bernstein in [3]).

IV. $F_{\rho,\varphi}[P_n] = \operatorname{Re} P_n^{(k)}(\rho e^{i\varphi})$, i.e.

$$\mu_i = 0, \quad i = 0, 1, \dots, k-1; \qquad \mu_i = \frac{i!}{(i-k)!}\rho^{i-k}\cos(i-k)\varphi,$$

$$i = k, k+1, \dots, n, \tag{5}$$

where $\rho > 0$, $\varphi \in [0, 2\pi]$. It suffices to consider $\varphi \in [0, \pi]$.

From inequalities (2) there follows the following assertion for the functional (5): whatever the numbers $\{t_i\}$ on $[0,1]$ and $\{y_i\}$ on the circle of radius ρ may be, we have

$$\max_{0\le\varphi\le 2\pi} N_F(\rho,\varphi) = \max_{0\le\varphi\le 2\pi} \sup_{P_n\in\mathcal{P}t} \operatorname{Re} P_n^{(k)}(\rho e^{i\varphi})$$

is attained at one of the polynomials $\pm M_n(x)$.

It can also be shown that on $[0,\pi]$, for $\rho \ge \rho_k$ (ρ_k is the greatest root of the polynomial $R_{n+1}^{(k)}(x)$; $\rho_k \le 1$), there are $n-k$ intervals $(\beta_m^{(k)}(\rho), \alpha_m^{(k)}(\rho))$, at whose points the extremal polynomials are polynomials different from $\pm M_n(x)$, and moreover

$$\beta_m^{(k)}(\rho), \alpha_m^{(k)}(\rho) \longrightarrow \frac{(2m-1)\pi}{2(n-k)} \quad (m = 1, 2, \dots, n-k).$$

These intervals are noteworthy in that the order of growth of $N_F^{(k)}(\rho,\varphi)$ (with respect to ρ) at each point of such an interval is equal to $n-k-1$, i.e. is one less than the order of growth of

$$\max_{0\le\varphi\le\pi} N_F^{(k)}(\rho,\varphi) = \left|M_n^{(k)}(-\rho)\right|.$$

V. $F_\xi[P_n] = \int_0^\xi P_n(x)\,dx$, i.e.

$$\mu_i = \frac{\xi^{i+1}}{i+1}, \quad i = 0, 1, \dots, n.$$

Consequently,

$$\left|\sup_{\mathcal{L}n\in\mathcal{P}_t} \int_0^\xi \mathcal{L}_n(x)\,dx\right| \le N_F(\xi) = \sum_{i=0}^{n} y_i \left|\frac{1}{R'_{n+1}(t_i)} \int_0^\xi R_{n+1,i}(x)\,dx\right|.$$

Denote by r the greatest of the roots of the polynomials

$$\left\{\int_0^\xi R_{n+1,i}(x)\,dx\right\}_{i=0}^{n}$$

for $-\infty < \xi < +\infty$. Obviously, $r \ge 0$. If $r = 0$, then at every point $\xi \ne 0$ of the real axis one of the polynomials $\pm M_n(x)$ is extremal. If $r > 0$, then the polynomial $M_n(x)$ (or $-M_n(x)$) is extremal at all points of the intervals $(-\infty, 0)$, $(r, +\infty)$. With respect to the interval $(0, r]$ we note that in each particular case–when a point $\xi_0 \in (0, r]$ and a set of points $\{t_i^{(0)}\}_{i=0}^{n}$ are prescribed–the determination of the extremal polynomial presents no difficulty.

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Note: Figure translations are in progress. See original paper for figures.

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