ON LOCALLY PERFECT MAPPINGS

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Abstract

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This paper considers locally perfect mappings, which stand in the same relation to perfect mappings as locally bicompact spaces do to bicompacta. All spaces are assumed to be T_1-spaces, and all mappings continuous.

A mapping $f : X \to Y$ will be called **locally perfect** if for each point x there exists a neighborhood U such that the image $f[U]$ of the closure U is closed in Y and $f|[U]$ is perfect.

Every mapping of a locally bicompact space is locally perfect. In particular, every mapping of a discrete space is locally perfect. Also, if $f : X \to Y$ is perfect and X_0 is an open set in X, then the restriction $f|X_0$ is locally perfect.

Let $f : X \to Y$ be locally perfect. Denote by Y^* the set of points $y \in Y$ at which f is not perfect (i.e., either f is not closed, or the full preimage $f^{-1}y$ is not bicompact).

Theorem 1. *Let $f : X \to Y$ be a locally perfect mapping. Then there exist a space $\widetilde{X} \supset X$ and a perfect mapping $\tilde{f} : \widetilde{X} \to Y$ such that: 1) $\tilde{f}|X = f$; 2) $\tilde{f}(\widetilde{X} \setminus X) = Y^*$; 3) the restriction $\tilde{f}|(\widetilde{X} \setminus X)$ is one-to-one; 4) X is dense in $\widetilde{X}$; 5) X is open in $\widetilde{X}$. These conditions determine $\widetilde{X}$ and $\tilde{f}$ uniquely.*

The space $\widetilde{X}$ is obtained from X as follows: to each full preimage of a point $y \in Y^*$ we add one new point, and to the full preimages of points $y \in Y \setminus Y^*$ we add nothing. We leave the topology on X unchanged. If $x \in \widetilde{X} \setminus X$, then its neighborhoods will be sets of the form $U = \tilde{f}^{-1}V \setminus F$, where $V = V_y$ is an arbitrary neighborhood of the point $y = \tilde{f}x$ in Y, $F \subset X$ and is closed in X, fF is closed in Y, and the mapping $f|F$ is perfect.

The mapping $\tilde{f}$ is closed at the points $y \in Y^*$. Indeed, let Q be open in $\widetilde{X}$, $\tilde{f}^{-1}y \subset Q$. There exists a neighborhood $U = \tilde{f}^{-1}V \setminus F$ of the point $x_0 = \tilde{f}^{-1}y \cap (\widetilde{X} \setminus X)$, contained in Q. The set $F_1 = F \setminus (Q \cap X)$ is closed in F, hence fF_1 is closed in Y, and $y \notin fF_1$. Therefore $V_1 = V \setminus fF_1$ is a neighborhood of the point y. It is not hard to see that $\tilde{f}^{-1}V_1 \subset Q$. The closedness of $\tilde{f}$ on $Y \setminus Y^*$ follows from the openness of this set in Y. It is easy to verify that the full preimages $\tilde{f}^{-1}y$, where $y \in Y^*$, are bicompact. Thus the mapping $\tilde{f}$ is perfect.

In what follows, for a locally perfect mapping $f : X \to Y$, the symbols $\widetilde{X}$ and $\tilde{f}$ will always denote the space and mapping discussed in the theorem.

Let, for example, $X = \{(x, y) : 0 \le x \le 1,\ 0 \le y < 1\}$, Y be the segment $[0, 1]$ of the abscissa axis, and let f be the vertical projection of X onto Y. The mapping f is locally perfect, although it is not closed at any point of the space Y. In this case

$$\widetilde{X} = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$$

and $\tilde{f}$ is the vertical projection onto Y.

Theorem 2. *Let $f : X \to Y$ be a locally perfect mapping. If Y is locally bicompact at the point y_0, then X is locally bicompact at each point $x \in f^{-1}y_0$. Consequently, if Y is a locally bicompact space, then X is also locally bicompact.*

However, under passage to the image under a locally perfect mapping, local bicompactness is not preserved.

Theorem 3. Let the mapping $f : X \to Y$ be locally perfect.
1) If X and Y are Hausdorff, then $\widetilde{X}$ is Hausdorff; 2) if X and Y are completely regular, then $\widetilde{X}$ is completely regular; 3) if X and Y have countable bases, then $\widetilde{X}$ has a countable base.

1) This is proved directly.

2) Let B be closed in $\widetilde{X}$ and $x_0 \notin B$, $\tilde{f}x_0 \notin \tilde{f}B$. Then the existence of a function continuous on $\widetilde{X}$, equal to zero at the point x_0 and to one on B, follows from the complete regularity of Y. If $y_0 = \tilde{f}x_0 \in \tilde{f}B$, $x_0 \in \widetilde{X} \setminus X$, then, since X is completely regular and the set $A = B \cap \tilde{f}^{-1}y_0$ is bicompact, there exists a neighborhood U, $[U]_{\widetilde{X}} \subset X$, of the set A and a function ψ, continuous on X, such that $\psi|_{X\setminus U} \equiv 0$ and $\psi \equiv 1$ in some smaller neighborhood W, $[W]_X \subset U$, of the set A. Then the function φ_1, $\varphi_1(x) = \psi(x)$ if $x \in U$, and $\varphi_1(x) = 0$ if $x \in \widetilde{X} \setminus U$, is continuous on $\widetilde{X}$, $\varphi_1(x_0) = 0$, and $\varphi_1|_W \equiv 1$. Since $B_1 = B \setminus W$ is closed in $\widetilde{X}$, and $y_0 \notin \tilde{f}B_1$, by what has been proved there exists a continuous function φ_2, $\varphi_2(x_0) = 0$, $\varphi_2|_{B_1} \equiv 1$. Then the function $\varphi = \max(\varphi_1, \varphi_2)$ is continuous on $\widetilde{X}$ and separates the point x_0 and the set B. Finally, if $x_0 \in X$ and $\tilde{f}x_0 \in \tilde{f}B$, then the assertion follows at once from the complete regularity of X.

3) Let $\{U_n\}$ be a countable base of X and $\{V_m\}$ a countable base of Y. Then a countable base of $\widetilde{X}$ is formed by $\{U_n\}$ and all sets of the form

$$U'_{m,k} = \tilde{f}^{-1}V_m \setminus \bigcup_{i=1}^{k} [U_i].$$

Corollary. Let $f : X \to Y$ be a locally perfect mapping of a Hausdorff space X onto a regular space Y. Then X is regular.

However, there exist examples of a locally perfect mapping of a completely regular, but not normal, space onto a normal space.

Theorem 4. If X and Y are metric spaces with countable bases and the mapping $f : X \to Y$ is locally perfect, then the space $\widetilde{X}$ is metrizable.

Without the assumption of a countable base, the theorem is false.

Example. The space X is the interval $[0, 1]$ with the discrete topology, Y is the same interval with the usual topology, and f is the identity mapping of X onto Y. Then f is locally perfect, the space X is metric, and Y is even compact. The extended space $\widetilde{X}$ is bicompact, but does not have a countable base, and therefore $\widetilde{X}$ is not metrizable.

Under locally perfect mappings, metrizability, generally speaking, is preserved neither under passage to the image nor under passage to the preimage.

Theorem 5. Let X and Y be metric spaces with countable bases, with Y an absolute G_δ, and let the mapping $f : X \to Y$ be locally perfect. Then X is also an absolute G_δ.

Indeed, the extended space $\widetilde{X}$ is an absolute G_δ ((1), Theorem 4), and since X is open in $\widetilde{X}$, X is also an absolute G_δ.

Theorem 6. Let X and Y be finite-dimensional metric spaces with countable bases and let the mapping $f : X \to Y$ be locally perfect. If $\dim X - \dim Y = k$, then there exists a point of the space Y whose full preimage has dimension $\geq k$.

Let $\tilde{f} : \widetilde{X} \to Y$ be a perfect extension of f. Since $\dim X \leq \dim \widetilde{X}$, there exists a point $y_0 \in Y$ such that $\dim \tilde{f}^{-1} y_0 \geq k$. Then also $\dim f^{-1} y_0 \geq k$.

In what follows, the spaces under consideration will be assumed Hausdorff.

Let f be a mapping of X onto Y. If $X_1 \supset X$, X is dense in X_1, and f_1 is a perfect mapping of X_1 onto Y which coincides with f on X, then f_1 will be called a perfect extension of the mapping f.

Theorem 7. Let $f : X \to Y$ be a locally perfect mapping and let $\tilde{f}_1 : \tilde{X}_1 \to Y$ be any perfect extension of it, with $\tilde{X}_1$ Hausdorff. Then: 1) X is open in $\tilde{X}_1$; 2) if $y \in Y \setminus Y^*$, then $\tilde{f}_1^{-1} y = f^{-1} y$, where Y^* is the set of points $y \in Y$ at which f is not perfect.

The proof is based on the following lemma.

Lemma. Let $\varphi : X \to Y$ be a perfect mapping. If $A \subset X$, $\varphi | A$ is perfect, and φA is closed in Y, then A is closed in X.

This follows easily from (2), Lemma 1.4.

Proof of Theorem 7. 1) Since f is locally perfect, each point x has in X a neighborhood U such that $[U]_X$ is closed in $\tilde{X}_1$ (by the lemma). There exists a neighborhood V_x in $\tilde{X}_1$ such that $U = X \cap V$. Then $(V \setminus [U]_X) \cap X = \Lambda$, and since X is dense in $\tilde{X}_1$, $V \setminus [U]_X$ is empty. Hence $V \subset [U]_X$, i.e. $V \subset X$, $U = V$.

2) Let $x_0 \in \tilde{f}_1^{-1}y \setminus f^{-1}y$, $y \in Y \setminus Y^*$. Since $y \in Y \setminus Y^*$, there exists a neighborhood $U[f^{-1}y]$ such that $[U]_X$ is closed in $\tilde{X}_1$, and a neighborhood V_y such that $f^{-1}V \Subset U$. $(\tilde{f}_1^{-1}V \setminus f^{-1}V) \cap X = \Lambda$, then $W = \tilde{f}_1^{-1}V \setminus [U]_X$ is nonempty, open in $\tilde{X}_1$, and $W \cap X = \Lambda$, which is impossible.

If the mapping $f : X \to Y$ is locally perfect, then the perfect extension $\tilde{f}$ and the space $\tilde{X}$ constructed in Theorem 1 are minimal in the following natural sense.

Theorem 8. Let $\tilde{f}_1 : \tilde{X}_1 \to Y$ be any perfect extension of the locally perfect mapping $f : X \to Y$. Then there exists a continuous mapping $\tilde{h} : \tilde{X}_1 \to \tilde{X}$ such that $\tilde{h}|x| = x$, if $x \in X$, and $\tilde{f}_1 = \tilde{f}\tilde{h}$.

Define $\tilde{h}$ as follows. If $x \in X$, then $\tilde{h}|x| = x$. If, however, $x \in \tilde{X}_1 \setminus X$, then, by Theorem 7, there exists a unique point $x' \in \tilde{X} \setminus X$ such that $\tilde{f}_1 x = \tilde{f}x'$, and then we set $\tilde{h}|x| = x' \in \tilde{X} \setminus X$. It is easy to see that $\tilde{h}$ is continuous on $\tilde{X}_1$.

Now we restrict ourselves to considering completely regular spaces. Every continuous mapping $f : X \to Y$ extends uniquely to a continuous mapping $f_B : \beta X \to BY$, where BY is any bicompact extension of the space Y. The mapping f_B is perfect on the set $f_B^{-1}Y$. Denote $f_B^{-1}Y$ by $\tilde{X}_0$; $\tilde{X}_0$ is completely regular. Let $f_B|\tilde{X}_0 = \tilde{f}_0$. The space X is contained in $\tilde{X}_0$ and is dense in it; hence the mapping $\tilde{f}_0 : \tilde{X}_0 \to Y$ is a perfect extension of the mapping $f : X \to Y$. The extension $\tilde{f}_0$ and the space $\tilde{X}_0$ are maximal, namely:

Theorem 9. Let $\tilde{f}_1 : \tilde{X}_1 \to Y$ be any perfect extension of the mapping $f : X \to Y$, where X, Y, and $\tilde{X}_1$ are completely regular. Then there exists a continuous mapping $\tilde{h} : \tilde{X}_0 \to \tilde{X}_1$ such that $\tilde{h}|x| = x$, if $x \in X$, and $\tilde{f}_0 = \tilde{f}_1\tilde{h}$.

We shall call a mapping $f : X \to Y$ **locally closed** if every point $x \in X$ has a neighborhood U such that $f[U]$ is closed in Y, the mapping $f|[U]$ is closed, and $[U] \cap \operatorname{Fr} f^{-1}y$ is a bicompactum for any point $y \in f[U]$.

Theorem 10. Let f be a locally closed mapping of a metric space X onto a space Y satisfying the first axiom of countability. Then there exist a space $\tilde{X} \supset X$ and a closed mapping $\tilde{f} : \tilde{X} \to Y$ such that: 1) $\tilde{f}|X = f$; 2) $\tilde{f}(\tilde{X} \setminus X) = Y^*$, where Y^* is the set of points $y \in Y$ at which f is not closed; 3) $\tilde{f}|(\tilde{X} \setminus X)$ is one-to-one; 4) X is dense in $\tilde{X}$; 5) X is open in $\tilde{X}$. The conditions 1, 2, and 3 determine $\tilde{X}$ and $\tilde{f}$ uniquely.

The space $\tilde{X}$ is obtained from X as follows: to each full inverse image of a point $y \in Y^*$ we add one new point, and to the full inverse images of points $y \in Y \setminus Y^*$ we add nothing. We leave the topology on X unchanged. If $x \in \tilde{X} \setminus X$, its neighborhoods will be the sets of the form $U = \tilde{f}^{-1}V \setminus F$, where $V = V_y$ is an arbitrary neighborhood of the point $y = \tilde{f}x$ in Y, $F \subset X$ and is closed in X, fF is closed in Y, the mapping $f|F$ is closed, and $F \cap \operatorname{Fr} f^{-1}y$ is bicompact for $y \in fF$.

Analogously to Theorem 1, the closedness of $\tilde{f}$ is proved.

If the space Y is Hausdorff (respectively regular), then $\tilde{X}$ is also Hausdorff (respectively regular).

Theorem 11. *Let f be a locally closed mapping of a normal space X onto a space Y with the first axiom of countability. Then the boundary of the full preimage $f^{-1}y$ of each point y is locally compact.*

If f is a locally perfect mapping of a metrizable space X onto a space Y with the first axiom of countability, then, extending it by the method described in Theorem 10, we obtain a closed but, generally speaking, not perfect extension.

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Note: Figure translations are in progress. See original paper for figures.