ON PARABOLIC SYSTEMS WITH SMALL PARAMETERS AT THE HIGHEST DERIVATIVES

1967

SovietRxiv

Abstract

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UDC 517.946

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ON PARABOLIC SYSTEMS WITH SMALL PARAMETERS AT THE HIGHEST DERIVATIVES

(Presented by Academician S. L. Sobolev on 6 VII 1966)

The paper studies questions connected with the limiting behavior of solutions of the Cauchy problem for parabolic systems as the small parameter at the highest derivatives tends to zero. First we obtain sufficient conditions for convergence, under such a limiting transition, of the solution of the parabolic system to the solution of the limiting equation.

Consider solutions of the linear system

$$A\mathbf{u}_t^\varepsilon + \sum_{i=1}^m B_i \mathbf{u}_{x_i}^\varepsilon + B_0 \mathbf{u}^\varepsilon = \varepsilon \left(\sum_{i,j=1}^m C_{ij} \mathbf{u}_{x_i x_j}^\varepsilon + \sum_{i=1}^m C_{i0} \mathbf{u}_{x_i}^\varepsilon + C_{00} \mathbf{u}^\varepsilon \right) + \mathbf{f} \quad (1)$$

with one and the same initial condition $\mathbf{u}_0(\mathbf{x})$. Here A, B_i, C_{ij} are $l \times l$ matrices; $\mathbf{u}, \mathbf{f}, \mathbf{u}_0$ are l-dimensional vectors.

Definition. System (1) **belongs to the class** $Q(r, M, T)$ if the matrices $A, A_t, B_0, B_i, C_{00}, C_{i0}, C_{ij}, (B_i)_{x_k}, (C_{ij})_{x_k}$, $i, j, k = 1, \dots, m$, and all their derivatives up to order r inclusive are uniformly continuous and bounded in norm by the constant M in the domain $0 \le t \le T$, $-\infty < x_1, \dots, x_m < \infty$.

For $\mathbf{y} = \{y_1, \dots, y_q\}$ set $\|\mathbf{y}\|^2 = \sum_{i=1}^q |y_i|^2$; for $\vec{\varphi}(t, \mathbf{x}) = \{\varphi_1(t, \mathbf{x}), \dots, \varphi_l(t, \mathbf{x})\}$ set

$$\|\vec{\varphi}(t)\|_{\lambda,n} = \sum_{q_1+\dots+q_m \le n} \int_{E_m} |\varphi^{(q_1,\dots,q_m)}|^2 \exp(2\lambda \|\mathbf{x}\|)\, dX,$$

$$\|\vec{\varphi}\|_{T,\lambda,n} = \sup_{0 \le t \le T} \|\vec{\varphi}(t)\|_{\lambda,n}.$$

Condition 1. $A = A'$, $B_i = B_i'$, $i = 1, \dots, m$;

$$(A\vec{\xi},\vec{\xi}) \geq q_1(\vec{\xi},\vec{\xi}), \qquad \sum_{i,j=1}^{m}(C_{ij}\vec{\xi}_i,\vec{\xi}_j) \geq q_2\sum_{i=1}^{m}(\vec{\xi},\vec{\xi}_i)$$

for all real $\vec{\xi},\vec{\xi}_i$, where $q_1, q_2 > 0$ do not depend on $(t, \mathbf{x})$.

This condition ensures the hyperbolicity of system (1) for $\varepsilon = 0$ and its evolution property for $\varepsilon > 0$.

Theorem 1. *Under Condition 1, in the cases: a) $\lambda = \lambda' = 0$, m arbitrary; b) $\lambda \leq \lambda'$, $m = 1$; c) $\lambda < \lambda'$, $1 < m$, for a system of the class $Q(r, m, T)$, when $n \leq r,\ k \leq 2$, the inequalities*

$$\|\mathbf{u}^0 - \mathbf{u}^\varepsilon\|_{T,\lambda,n-k} \leq \sigma_k(n, M, T, q_1, q_2, \lambda, \lambda')(\|\mathbf{u}_0\|_{\lambda',n} + \|\mathbf{f}\|_{T,\lambda',n})\varepsilon^{k/2}.$$

For $\lambda = \lambda' = 0$ the theorem is proved by obtaining uniform in ε estimates of the norms $\|\mathbf{u}^\varepsilon\|_{T,0,n}$ and subsequent estimates of the norms of the necessary quantities. In the other cases, for $m = 1$ we reduce the problem to the case where $\mathbf{u}$ and $\mathbf{f}$ are nonzero only in the direction of one of the half-axes, while for $m > 1$ only in certain cones of small opening. Then, by a change of variables of the form $\mathbf{u} = \exp\left(\sum_{i=1}^m d_i x_i\right)\mathbf{v}$, we reduce the problem to the case $\lambda = \lambda' = 0$.

Theorem 2. Let $\mathbf{u}_0, \mathbf{f}$ be measurable,

$$\|\mathbf{u}_0\| + \|\mathbf{f}\| \leq \exp(a\|\mathbf{x}\|) \quad \text{for } \|\mathbf{x}\| \geq R_2,$$

$$\int_{\|\mathbf{x}\|\leq R_2}\left(\|\mathbf{u}_0\|^2 + \int_0^T \|\mathbf{f}\|^2\,dt\right)dX < \infty.$$

Then for a system of the class $Q(1, T, M)$, for any R_1, uniformly for $0 \leq t \leq T$, we have

$$\lim_{\varepsilon\to 0}\int_{\|\mathbf{x}\|\leq R_1}\|\mathbf{u}^\varepsilon(t,\mathbf{x}) - \mathbf{u}^0(t,\mathbf{x})\|^2\,dX = 0.$$

This is proved with the aid of Theorem 1 by approximating $\mathbf{u}_0$ and $\mathbf{f}$ by smooth functions.

Consider the case of a system with constant coefficients.

Theorem 3. In order that the solutions of equation (1) converge to solutions of the limiting equation for all $\mathbf{u}_0$ and $\mathbf{f}$ satisfying the condition $\|\mathbf{u}_0\|_q + \|\mathbf{f}\|_{T,0,q} < \infty$ for some $q < \infty$, it is necessary that condition 2 be satisfied.

Condition 2. For real $\lambda_1, \ldots, \lambda_m$, among the roots of the equation

$$\left|\det\left\|A\mu + i\sum_{k=1}^{m} B_k\lambda_k + \sum_{k,j=1}^{m} C_{kj}\lambda_k\lambda_j\right\|\right| = 0 \tag{2}$$

there are no roots μ with positive real part.

The validity of the theorem follows from the existence for (1) of particular solutions of the form $c\exp\left(\left((\mu + o(1))t + i\sum_{k=1}^{m}\lambda_k x_k\right)/\varepsilon\right)$.

Condition 2 is stronger than the condition of evolutionarity of system (1).

Condition 3. For real $\lambda_1, \dots, \lambda_m$, $\sum_{k=1}^{m}\lambda_k^2 \neq 0$, the roots of equation (2) satisfy the condition

$$\operatorname{Re}\mu \leq -q_3\sum_{k=1}^{m}\lambda_k^2 < 0.$$

In what follows we consider the case $m = 1$. We give a number of interesting conditions supplementing the theorems proved.

Lemma 1. Let $A = E$, B_1 be diagonal, and let its diagonal elements be ordered:

$$b_{11} = \dots = b_{l_1 l_1} < b_{l_1+1, l_1+1} = \dots = b_{l_2 l_2} < \dots = b_{l_t l} t.$$

In order that condition 2 be satisfied, it is necessary that the principal minors of the matrix C_{11} corresponding to the sets of indices $(1, \dots, l_1)$, $(l_1 + 1, \dots, l_2)$, ..., have no eigenvalues with negative real parts.

Lemma 2. Let $A = C_{11} = E$, and let the roots of the equation

$$|\det\|B(t,x) - \mu B(t,x)\|| = 0$$

be simple and let condition 3 be satisfied at the point (t,x). Then, for $l = 2$, in a neighborhood of the point (t,x) the system is reducible to a form satisfying condition 1.

Lemma 3. Let

$$|\det\|A(t,x)\|| \neq 0,$$

and let the roots of the equation

$$|\det\|A(t,x) - \mu B(t,x)\|| = 0$$

be simple and let condition 3 be satisfied at the point (t,x). Then, for $l = 2$, in a neighborhood of the point (t,x) the system is reducible to a form satisfying condition 1.

For the study of the stability of discontinuous solutions of systems of quasilinear equations

$$\partial\varphi_i(\mathbf{u})/\partial t + \partial\psi_i(\mathbf{u})/\partial x = \varepsilon \sum_{j=1}^{l} \partial(c_{ij}(\mathbf{u})\,\partial u_i/\partial x)/\partial x \tag{3}$$

it is of interest ([1]) to find particular solutions of these equations in the form of a traveling wave: $u_i = z_i((x - \omega t)/\varepsilon)$, such that the limits $\lim_{y\to\pm\infty} z_i(y) = z_i^{\pm}$ exist and are finite.

Substituting the solutions in this form into (3), we obtain that the functions z_i are solutions of the system

$$C_i - \omega\varphi_i(z) + \psi_i(z) = \sum_{j=1}^{l} c_{ij}(z)\frac{dz_j}{dx}, \tag{4}$$

which enter the singular points $Z_{\pm}(z_1^{\pm}, \dots, z_l^{\pm})$.

The type of the singular point Z in the linear approximation is determined by the number of positive and negative roots of the equation

$$\left|\det\left\|c_{ij}(Z)\mu + \omega\frac{\partial\varphi_i(Z)}{\partial u_j} - \frac{\partial\psi_i(Z)}{\partial u_j}\right\|\right| = 0. \tag{5}$$

Let, for $\varepsilon = 0$, system (3) be hyperbolic. The directions of the characteristics for $u = Z$ are determined from the equation

$$\left|\det\left\|-\rho\frac{\partial\varphi_i(Z)}{\partial u_j} + \frac{\partial\psi_i(Z)}{\partial u_j}\right\|\right| = 0. \tag{6}$$

Theorem 4. *Let $u = Z$ be a singular point of system (4); suppose that for $u = Z$ the system of equations in variations for system (3) satisfies condition 3; suppose that σ roots of equation (6) are less than ω, t roots are greater than ω, and the remaining $l - \sigma - t$ roots are equal to ω. Then equation (5) has t roots with positive real parts, σ with negative real parts, and $l - \sigma - t$ equal to 0.*

In the proof one counts the change in the number of zeros of equation (5) in the right and left half-planes as ω varies. In doing so the following is used: under condition 3, equation (5) cannot have roots $\mu = ib$ for $b \neq 0$, and the root $\mu = 0$, moreover simple, can occur only in the case when ω is a root of (6). In the case when the system of equations in variations satisfies the weaker condition 2, one may assert that equation (5) has no more than t roots with positive real parts and no more than σ with negative real parts.

Now suppose that for $Z = Z^-$ equation (6) has σ_- roots not less than ω, and for $Z = Z^+$ it has σ_+ roots not greater than ω. Then it follows from Theorem 5 that the manifold of solutions of (4) connecting the points Z^- and Z^+ has, generally speaking, dimension not greater than $\sigma_+ + \sigma_- - l - 1$. Thus, for $\sigma_+ + \sigma_- \leq l$, the existence of such solutions under condition 3 is an exceptional case. In particular, for the example of "nonuniqueness" constructed in the work (2), condition 2 is not fulfilled for the system of equations in variations.

Consider the system of equations

$$u^i + ((u^i)^2/2)_x = \varepsilon(b_{i1}u^1 + b_{i2}u^2)_{xx} \qquad (i = 1, 2). \tag{7}$$

$\mathbf{u} = \{u^1, u^2\}$ is called a generalized solution of this system for $\varepsilon = 0$ if, along any closed contour,

$$\oint u^i dx - \frac{1}{2}(u^i)^2 dt = 0, \tag{8}$$

$$\Delta u^i/\Delta x < c/t \qquad \text{for } t > 0. \tag{9}$$

Let

$$b_{11},\ b_{22},\ b_{11}b_{22} - b_{12}b_{21} > 0; \tag{10}$$

then condition 3 is fulfilled for the equations in variations.

Theorem 5 (N. N. Kuznetsov). *Let* $\max b_{12}, b_{21} > 0$. *Then system (7) has a solution of traveling-wave type which, as* $\varepsilon \to 0$, *converges to a certain function* $u(x/t)$ *satisfying relation (8), but not satisfying relation (9).*

For $0 \leq t \leq T$ we shall call $\mathbf{u}(t, x)$ an η-solution of the system $\mathcal{L}(\mathbf{u}) = 0$ with initial condition $\mathbf{u}_0$, if

$$\int_{-\infty}^{\infty} \left(\|\mathbf{u}(0, x) - \mathbf{u}_0(x)\| + \int_0^T \|\mathcal{L}(\mathbf{u})\| \, dt \right) dx \leq \eta.$$

Consider the initial conditions

$$u^i = u^i_+ \quad \text{for } x \geqslant 0; \qquad u^i = u^i_- \quad \text{for } x < 0,$$

such that system (7) for $\varepsilon = 0$ has a solution of the form of a shock wave.

Theorem 6. *Under condition (10), for the set of such quadruples* $(u^1_+, u^1_-, u^2_+, u^2_-)$, *with the exception of a certain manifold of lower dimension, the following alternative holds: either system (7) has a solution of traveling-wave type that tends to this shock wave as* $\varepsilon \to 0$; *or, for each* $\varepsilon > 0$, *one can indicate a certain* $C\varepsilon$-*solution* $\mathbf{u}^\varepsilon(t, x)$ *of system (7) such that*

$\lim_{\varepsilon\to 0}\mathbf{u}^\varepsilon(t,x)=\mathbf{u}(x/t)$*, where* $\mathbf{u}(x/t)$ *satisfies relation (8) and does not satisfy relation (9); the quantity* C *is determined by the initial conditions.*

In the proof of the theorem, examples of "nonuniqueness" constructed by N. N. Kuznetsov (Theorem 5) were used.

It is possible that the assertion of Theorem 6 indicates the essential nature of the condition for the existence of a traveling-wave solution for the stability of a shock wave with respect to the addition of a small "viscosity" in the case of convex equations of state.

We note that for the system of equations considered in works ([3,4]), condition 3 for the system of equations in variations is satisfied.

The author expresses gratitude to S. K. Godunov and N. N. Kuznetsov for valuable discussions.

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Received
29 VI 1966

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Note: Figure translations are in progress. See original paper for figures.