Investigation of a mixed problem for a class of third-order quasilinear differential equations

Authors: G. A. Rasulova

Date: 1967-01-01T00:00:00+00:00

Abstract

In the peer-reviewed work, the following one-dimensional mixed problem is investigated:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^3 u}{\partial t \partial x^2} = \lambda F[t, x, U, U_x, U_t, U_{xx}, U_{tx}], \tag{1}$$

$$U(t,0) = U(t,\pi) = 0, \quad U(0,x) = \varphi(x) \quad U_t(0,x) = \psi(x), \tag{2}$$

where $0 \le x \le \pi$, $0 \le t \le T < \infty$, $\alpha > 0$ is a fixed number; λ is a parameter; F, φ, ψ are given functions. In the work, local (i.e., valid for sufficiently small values of $|\lambda|$) and nonlocal existence and uniqueness theorems for the generalized solution, the solution almost everywhere, and the classical solution of problem A are proven. Local existence theorems are proven using the fixed-point principles of M. A. Krasnoselskii and Schauder, while nonlocal existence theorems are proven using the method of successive approximations and the reinforced Schauder principle. The continuous dependence of all three types of solutions to problem A on the initial data and on the right-hand side of equation (1) is studied. Furthermore, the boundedness and behavior as $t \to \infty$ of the solutions to problem A and their certain derivatives are studied when these solutions exist in the domain $0 \le x \le \pi$, $0 \le t < \infty$. Bibliography: 2 items.

Full Text

Preamble

This section considers the boundary value problem for a nonlinear partial differential equation of the form:

$$\frac{\partial^2 U}{\partial t^2} + a \frac{\partial^2 U}{\partial x^2} = \lambda F[t, x, U, U_x, U_t, U_{xx}, U_{tx}]$$

subject to the boundary conditions $U(t,0) = U(t,\pi) = 0$ and the initial conditions $U(0,x) = \phi(x)$, $U_t(0,x) = \psi(x)$, where $a > 0$ and $0 < x < \pi, 0 < t < T$.

Here, F, ϕ, and ψ are given functions satisfying specific smoothness requirements. We seek a solution $U(t,x)$ in the domain D_T that satisfies the governing equation (1) and the conditions (2).

1. Series Representation and Convergence

The solution $U(t,x)$ is sought in the form of a Fourier series:

$$U(t,x) = \sum_{n=1}^{\infty} U_n(t) \sin nx$$

where the coefficients $U_n(t)$ are functions of time. We define the norm in the space B^0 as $\|U\| = \sum_{n=1}^{\infty} \max_{0 \le t \le T} |U_n(t)|$. For $v_0 \ge 1$, we consider the convergence of these series and their derivatives. If the coefficients satisfy $|U_n| < \infty$, the series defines a continuous function. Specifically, for any $\epsilon > 0$, there exists an $N(\epsilon)$ such that the remainder of the series is bounded by ϵ, ensuring the uniform convergence of $U(t,x)$ in the specified domain.

2. Integral Equations for Coefficients

By substituting the series representation into the original differential equation and applying the boundary conditions, we derive a system of integro-differential equations for the coefficients $U_n(t)$:

$$U_n(t) = a_n(t)\phi_n + b_n(t)\psi_n + \lambda \int_0^t G_n(t-\tau) F_n(\tau) d\tau$$

where $a_n(t)$ and $b_n(t)$ are fundamental solutions related to the linear part of the operator, and $G_n(t)$ is the corresponding Green’ s function. The explicit forms of these coefficients depend on the relationship between the parameter a and the mode number n. Specifically, for $n > 2/a$, the solutions involve exponential decays, while for $n < 2/a$, they exhibit oscillatory behavior characterized by trigonometric functions.

3. Existence and Uniqueness Theorems

Theorem 1. Suppose the initial data $\phi(x)$ and $\psi(x)$ belong to $L_2[0,\pi]$ and satisfy the compatibility conditions $\phi(0) = \phi(\pi) = 0$. If the nonlinear function F satisfies a Lipschitz condition with respect to its arguments $U, U_x, U_t, U_{xx}, U_{tx}$ in a bounded domain, then for sufficiently small $|\lambda|$, there exists a unique solution $U(t,x)$ to the problem (1)-(2) in the space $B_{2,1}$.

The proof utilizes the contraction mapping principle. We define an operator P such that $U_{k+1} = P(U_k)$. By showing that P maps a closed ball in the functional space into itself and is a contraction, we guarantee the convergence of the iterative process $U_k \to U$. The error estimate for the k-th approximation is given by:

$$\|U_k - U\| \le \frac{M|\lambda|^k}{1-\alpha}$$

where $\alpha < 1$ is the contraction constant.

4. Continuous Dependence and Stability

The solution $U(t, x)$ depends continuously on the initial data ϕ, ψ and the nonlinear term F. If we consider a perturbed problem with data $\bar{\phi}, \bar{\psi}, \bar{F}$, the difference between the solutions satisfies:

$$\|U - \bar{U}\| \leq C \left(\|\phi - \bar{\phi}\| + \|\psi - \bar{\psi}\| + \|F - \bar{F}\| \right) \exp \left(\int_0^T C_2(\tau) d\tau \right)$$

This inequality ensures the stability of the solution with respect to small perturbations in the input parameters.

5. Asymptotic Behavior

As $t \to \infty$, the behavior of the solution is governed by the damping properties of the operator. Under the assumption that the integral of the nonlinear term F is bounded over the infinite domain, we can show that the solution $U(t, x)$ and its derivatives U_x, U_t tend to zero or remain bounded depending on the specific form of the dissipation in the system. Specifically, if the coefficients $a_n(t)$ and $b_n(t)$ decay exponentially, the total energy of the system dissipates over time.

6. Generalizations to Higher-Order Derivatives

The analysis can be extended to cases where F depends on higher-order spatial derivatives such as U_{xxx} or mixed derivatives U_{ttx}. In these instances, the requirements on the smoothness of the initial data $\phi(x)$ and $\psi(x)$ must be increased accordingly (e.g., requiring $\phi \in W_2^3[0, \pi]$). The existence of solutions in these higher-order Sobolev spaces is established using similar fixed-point arguments, provided the nonlinearities satisfy appropriate growth conditions.

Note: Figure translations are in progress. See original paper for figures.

Source: RussiaRxiv —Machine translation. Verify with original.