A REFINEMENT OF A THEOREM OF BASS

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A REFINEMENT OF A THEOREM OF BASS

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Let A be a local (commutative) Noetherian ring with identity satisfying the following conditions: a) the Krull dimension of the ring A is not greater than 1; b) A has no nilpotent elements; c) the integral closure of the ring A is a finitely generated A-module.

In § 7 of Bass' s paper ([1]), conditions are considered under which every indecomposable finitely generated torsion-free A-module is isomorphic to an ideal of the ring A. This question is of interest, in particular, from the point of view of the theory of integral representations. The methods developed in ([1]) are very interesting; however, there is an inaccuracy in the formulation of the result, and correcting it is the aim of the present paper.

From the restrictions imposed on the ring A it follows that the ring is a subring of a direct sum of local Dedekind rings

$$A \subset \bigoplus_{i=1}^{l} A_i.$$

We shall say that the ring A **satisfies condition** α (or, for brevity, is an α**-ring**) if every finitely generated, indecomposable, torsion-free (torsionless) A-module (i.e., a module that can be embedded in a free module) is isomorphic to an ideal of the ring A. In what follows, speaking of modules, we shall have in mind finitely generated torsion-free modules.

It follows from Day' s work ([2]) that for every α-ring $l \leq 3$. In ([1]) it is proved that for $l = 1, 2$ the ring A satisfies condition α if and only if every ideal of the ring A has at most two generators. Bass further asserts that for $l = 3$ the ring A satisfies condition α if and only if A is a triad of Dedekind rings. By a triad T of Dedekind rings A_1, A_2, A_3 with isomorphic residue field k and given homomorphisms $\varepsilon_i : A_i$ onto k, Bass calls the set of triples (a_1, a_2, a_3), $a_i \in A_i$, for which $a_1\varepsilon_1 = a_2\varepsilon_2 = a_3\varepsilon_3$. However, there exist (for $l = 3$) α-rings that are not triads. For example, the integral group ring of the cyclic group of order 4 is an α-ring ([3]) and is not a triad. We shall give a certain description of all α-rings (for $l = 3$) that are not triads.

Thus, let A be an α-ring for which $l = 3$, $A \subset A_1 \oplus A_2 \oplus A_3$. Denote, following ([1]), $\mathfrak{P}_i = A \cap (A_j \oplus A_k)$; $D_i = A \cap A_i = \mathfrak{P}_j \cap \mathfrak{P}_k$; $\mathfrak{M}$ is the maximal ideal of the ring A; π_i is a prime element of A_i; $k \simeq A/\mathfrak{M} \simeq A_i/\pi_{iA}i$ is the residue field, $i \neq j \neq k$; $i, j, k = 1, 2, 3$.

In ([1]) it is proved (and we shall use this) that if A is an α-ring ($l = 3$), then:

1) $A/\mathfrak{P}_i = A_i$;
2) $\mathfrak{P}_1 + \mathfrak{P}_2 = \mathfrak{P}_1 + \mathfrak{P}_3 = \mathfrak{P}_2 + \mathfrak{P}_3 = \mathfrak{M}$.

From 1) it follows, in particular, that D_i (as an A-module) is isomorphic to A_i, and also that A is a subring of some triad T.

Using 1), 2), it is not difficult to show that

$$\mathrm{Ext}(A_i, A_j) \simeq k.$$

Denote $\overline{\mathfrak{P}}_j = A/A \cap A_j$, $j = 1, 2, 3$. Obviously, $\mathrm{Ext}(\overline{\mathfrak{P}}_j, A_j)$ either is cyclic and (since $\overline{\mathfrak{P}}_j$ is an extension of A_i by A_k) is isomorphic to k, or is an extension of k by k. In the latter case

$$\mathrm{Ext}(\overline{\mathfrak{P}}_j, A_j) \cong A_j/\pi_j^2 A_j, \quad j = 1, 2, 3.$$

Let X be an arbitrary A-module. Put $X_i = \{x, xD_i = 0\}$, $\overline{X}_i = X/X_i$. $\overline{X}_i$ is an A_i-module, since $\overline{X}_i \mathfrak{P}_i = 0$. X_i is a $\mathfrak{P}_i$-module, since $X_{iD}i = 0$. It is not difficult to verify that $\overline{X}_i$ is a finitely generated torsion-free A_i-module, and X_i is a finitely generated torsion-free $\mathfrak{P}$-module. Since every finitely generated torsion-free module over a local Dedekind ring is isomorphic to a direct sum of rings, we have $\overline{X}_i = A_i^{(n)}$ ($A_i^{(n)}$ is the direct sum of n copies of A_i). Using the fact that $\mathrm{Ext}(A_j, A_k) \cong k$, it is not difficult to show that

$$X_i + \overline{\mathfrak{P}}_i^{(m_1)} \oplus A_j^{(m_2)} \oplus A_k^{(m_3)}, \quad i \neq j \neq k.$$

Consequently,

$$\mathrm{Ext}(\overline{X}_i, X_i) = \mathrm{Ext}(A_i, \overline{\mathfrak{P}}_i)^{(nm_1)} \oplus \mathrm{Ext}(A_i, A_j)^{(nm_2)} \oplus \mathrm{Ext}(A_i, A_k)^{(nm_3)}.$$

Using the technique customary in the theory of integral representations (see, for example, (4)), one can show that if $\mathrm{Ext}(A_i, \overline{\mathfrak{P}}_i) \cong k$, then A has 8 indecomposable modules; if, however, $\mathrm{Ext}(A_i, \overline{\mathfrak{P}}_i) \cong A_j/\pi_j^2 A_j$, then A has 9 indecomposable modules. In both cases all these modules are isomorphic to ideals of the ring A. $A \subseteq T$, and hence every T-module is an A-module. The triad has 8 indecomposable modules (1). If $A \subset T$, then A must have at least 9 indecomposable modules (8 T-modules and A itself). Thus, if A has 8 indecomposable modules, then A is a triad.

Suppose now that A is not a triad and, consequently, has 9 indecomposable modules. $A \subset T$. We shall show that A is a maximal submodule of the triad T, considered as an A-module. A is a ring with identity, and hence it cannot be contained in any T-submodule of the triad T. Since every A-module is a direct sum of a T-module and of the module $A^{(n)}$, every module lying between A and T would have to be isomorphic to A. In this case the ring A would be a maximal submodule of some module isomorphic to A. Since the module A has one maximal submodule, it would follow that A is a principal-ideal ring, which of course cannot be the case, if only because $l = 3$. From the same considerations it follows that the maximal ideal $\mathfrak{M}$ of the ring A is a T-submodule. It is not difficult to show that if $\mathfrak{M}$ were decomposable, then A would be a triad. Consequently, $\mathfrak{M}$ is a maximal indecomposable T-ideal in the maximal ideal $\mathfrak{M}(T)$ of the triad T.

Simple computations show that in $\mathfrak{M}$ one can choose three generators: $(\pi_1^2, 0, 0)$; $(\pi_1 X_2, \pi_2, 0)$; $(\pi_1 X_3, 0, \pi_3)$, where X_2, X_3 are elements of the ring A_1 not lying in the prime ideal. In other words,

$$\mathfrak{M} = \{(\pi_1 a_1, \pi_2 a_2, \pi_3 a_3),\ \overline{a}_1 = \overline{a}_2 X_2 + \overline{a}_3 X_3\},$$

where $\overline{a}_1, \overline{a}_2, \overline{a}_3$ are elements of the residue field k to which the elements a_1, a_2, a_3 pass under the ring homomorphisms A_1, A_2, A_3 onto k fixed in the definition of the triad; X_2, X_3 are certain nonzero elements of k.

Thus, the ring A must lie between T and $\mathfrak{M}$, with T/A isomorphic to $A/\mathfrak{M}$ as an A-module. From the form of $\mathfrak{M}$ we conclude that if such a ring A exists, then it satisfies conditions 1), 2) and, consequently, as we have shown, all its indecomposable modules are isomorphic to ideals. Therefore, in order to describe all α-rings (for $l = 3$) that are not triads, it remains to check for which A_1, A_2, A_3, X_2, X_3 there exists a ring A such that $T/A \cong A/\mathfrak{M}$, in other words, in what case the factor ring $\overline{T} = T/\mathfrak{M}$ contains a subfield $\overline{A}$ such that $(\overline{T} : \overline{A}) = 2$. Note that $\overline{T}$ has one ideal $\mathfrak{M}(T)$.

Lemma. Let Λ be a commutative ring with identity with a unique nontrivial ideal $I \neq p\Lambda$. Then Λ contains a subfield $\bar{k}$ such that

$$(\Lambda : \bar{k}) = 2.$$

Let the characteristic of $k = \Lambda/I$ be p. It is not difficult to show that the set of p-th powers forms a subfield k' of the ring Λ. Using Zorn' s lemma, choose a maximal field $\bar{k}$ among the subfields of the ring Λ containing k'. Let φ be the natural mapping of Λ onto k. Suppose that $\varphi(\bar{k}) \neq k$. Then there is a simple extension of the field $\varphi(\bar{k})$ contained in k. Considering separately the cases where this extension is transcendental, algebraic separable, and purely inseparable, one can show in each case that $\bar{k}$ is not maximal. In the last case one must use the fact that $\bar{k} \supset k'$. Hence $\varphi(\bar{k}) = k$, $(\Lambda : \bar{k}) = 2$. In the case where the characteristic of k is 0, the proof is simplified because of the absence of inseparable extensions.

We also note that in the case where k is a finite field, $k = k'$, and therefore $\bar{k}$ is determined uniquely.

Thus, if the characteristic of k is 0, there always exists A such that

$$T/A \cong A/\mathfrak{M}.$$

If, however, the characteristic of k is p, then for the existence of such a ring it is necessary and sufficient that

$$\mathfrak{M}(T) \neq pT,$$

i.e. that

$$(p, p, p) \in \mathfrak{M}.$$

Let the characteristic of k be p. Consider the ideal pA_i in A_i. If the characteristic of A_i is 0, then $pA_i \neq 0$ and there is a number s_i such that

$$pA_i = \pi_i^{s_i} A_i.$$

If the characteristic of A_i is p, then we agree to regard s_i as equal to ∞. In the case $s_i = 1$ we shall regard $\pi_i = p$. We agree that

$$s_1 \geq s_2 \geq s_3.$$

Simple calculations show that, for $s_i \geq 2$ $(i = 1, 2, 3)$, a ring A with the properties we require exists for arbitrary $\bar{X}_2, \bar{X}_3$; for $s_1 \geq 2,\ s_2 \geq 2,\ s_3 = 1$ no such ring exists for any $\bar{X}_2, \bar{X}_3$; for $s_1 \geq 2,\ s_2 = s_3 = 1$ the ring A exists when

$$\bar{X}_2 + \bar{X}_3 = 0;$$

for

$$s_1 = s_2 = s_3 = 1$$

the ring exists when

$$\bar{X}_2 + \bar{X}_3 = 1.$$

In the last case, for $p = 2$, such a ring obviously does not exist ($\bar{X}_2, \bar{X}_3$ are distinct from zero) for any $\bar{X}_2, \bar{X}_3$; when the characteristic is different from 2, the corresponding $\bar{X}_2, \bar{X}_3$ can, of course, be chosen.

Thus, if A_1, A_2, A_3 are three local Dedekind rings with isomorphic residue field, then, except for the case where one of the ramification coefficients is equal to 1 and the others are not equal to 1, and the case where all ramification coefficients are equal to 1 and the characteristic of the residue field is 2, in the direct sum

$$A_1 \oplus A_2 \oplus A_3$$

there are α-rings distinct from triads. Every such ring can be obtained in the following way: take a certain triad T of the given rings A_1, A_2, A_3 and construct in it a submodule

$$\mathfrak{M} = \{(\pi_1 a_1,\ \pi_2 a_2,\ \pi_3 a_3),\ \bar{a}_1 = \bar{a}_2 \bar{X}_2 + \bar{a}_3 \bar{X}_3\},$$

where $\bar{X}_2, \bar{X}_3$ are some nonzero elements of the residue field, with

$$\bar{X}_2 + \bar{X}_3 = 1$$

when $s_1 = s_2 = s_3$, and

$$\bar{X}_2 + \bar{X}_3 = 0$$

when $s_1 \geq 2,\ s_2 = s_3 = 1$.

If k is a finite field, then

$$A = T^p + \mathfrak{M}.$$

If k is an arbitrary field, then choose in the factor ring

$$\bar{T} = T/\mathfrak{M}$$

some subfield $\bar{k}$ such that

$$(\bar{T} : \bar{k}) = 2.$$

Then A is the full preimage of $\bar{k}$ under the natural homomorphism

$$T \to \bar{T}.$$

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1. H. Bass, *Math. Zs.*, **82**, No. 1, 8 (1963).

2. E. C. Dade, *Ann. Math.*, **77**, No. 2 (1963).

3. A. V. Roiter, *Vestn. Leningrad. Univ.*, No. 19, 65 (1960).

4. A. Heller, J. Reiner, *Ann. Math.*, **76**, No. 1 (1962).

Note: Figure translations are in progress. See original paper for figures.