Expansion of an arbitrary function in an integral with respect to the squares of Legendre functions with complex index

N. N. Lebedev

1967

SovietRxiv

Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

1967, Vol. III, No. 3 UDC 517.564.4

EXPANSION OF AN ARBITRARY FUNCTION INTO AN INTEGRAL OVER SQUARES OF LEGENDRE FUNCTIONS WITH COMPLEX INDEX

In the study of many problems in mathematical physics, a significant role is played by the representation of an arbitrary function $f(x)$ defined on a given interval as a Mehler-Fock integral:

$$f(x) = \int_0^\infty \tau \tanh(\pi\tau) P_{-\frac{1}{2}+i\tau}(x) d\tau \int_1^\infty f(y) P_{-\frac{1}{2}+i\tau}(y) dy$$

This expansion is fundamental for solving boundary value problems in coordinate systems where the boundaries are surfaces of constant values in conical or toroidal coordinates. The Legendre functions of complex index, $P_{-\frac{1}{2}+i\tau}(x)$, form the basis of this integral transform, which generalizes the Fourier transform to cases involving spherical symmetry and hyperbolic geometries.

The validity of such expansions depends on the properties of the function $f(x)$ and the asymptotic behavior of the Legendre functions. In this paper, we investigate the conditions under which an arbitrary function can be decomposed into an integral involving the squares of these Legendre functions. Such representations are particularly useful when dealing with quadratic forms of fields or energy densities in physical applications.

$1 < X < \infty$,

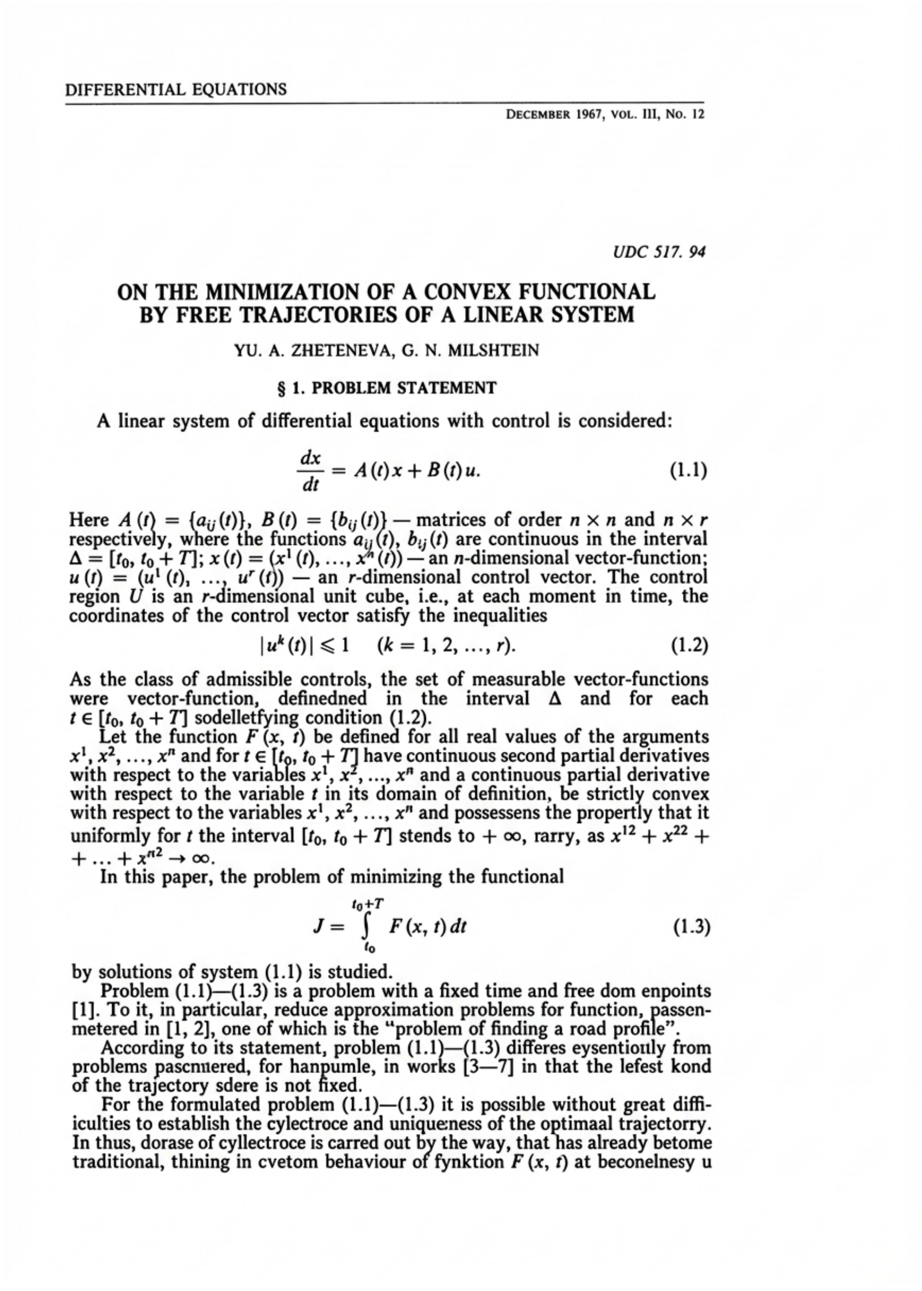
where P is the spherical Legendre function of the first kind. The objective of the present work is to derive a formula of a similar type, but one that contains the squares of Legendre functions. In this paper, the following theorem is proven.

Let $f(x)$ be an arbitrary function defined on a given interval and satisfying the conditions $f(x) \in L(1, a)$ and $f(x) \in ....$ Then the following expansion holds:

$$f(x) = ...$$

$1 < X < \infty$,

(z) is the Legendre function of the first kind. The integrals with respect to the variable in the left and right sides of equation (3) are Lebesgue integrals,



DIFFERENTIAL EQUATIONS

DECEMBER 1967, VOL. III, NO. 12

UDC 517. 94

ON THE MINIMIZATION OF A CONVEX FUNCTIONAL BY FREE TRAJECTORIES OF A LINEAR SYSTEM

YU. A. ZHETENEVA, G. N. MILSHTEIN

§ 1. PROBLEM STATEMENT

A linear system of differential equations with control is considered:

$$\frac{dx}{dt} = A(t)x + B(t)u. \tag{1.1}$$

Here $A(t) = \{a_{ij}(t)\}$, $B(t) = \{b_{ij}(t)\}$ — matrices of order $n \times n$ and $n \times r$ respectively, where the functions $a_{ij}(t)$, $b_{ij}(t)$ are continuous in the interval $\Delta = [t_0, t_0 + T]$; $x(t) = (x^1(t), \ldots, x^n(t))$ — an n-dimensional vector-function; $u(t) = (u^1(t), \ldots, u^r(t))$ — an r-dimensional control vector. The control region U is an r-dimensional unit cube, i.e., at each moment in time, the coordinates of the control vector satisfy the inequalities

$$|u^k(t)| \leqslant 1 \quad (k = 1, 2, \ldots, r). \tag{1.2}$$

As the class of admissible controls, the set of measurable vector-functions were vector-function, definedned in the interval Δ and for each $t \in [t_0, t_0 + T]$ sodelletfying condition (1.2).

Let the function $F(x, t)$ be defined for all real values of the arguments $x^1, x^2, \ldots, x^n$ and for $t \in [t_0, t_0 + T]$ have continuous second partial derivatives with respect to the variables $x^1, x^2, \ldots, x^n$ and a continuous partial derivative with respect to the variable t in its domain of definition, be strictly convex with respect to the variables $x^1, x^2, \ldots, x^n$ and possessens the propertly that it uniformly for t the interval $[t_0, t_0 + T]$ stends to $+\infty$, rarry, as $x^{12} + x^{22} + \ldots + x^{n2} \to \infty$.

In this paper, the problem of minimizing the functional

$$J = \int_{t_0}^{t_0+T} F(x, t)\, dt \tag{1.3}$$

by solutions of system (1.1) is studied.

Problem (1.1)—(1.3) is a problem with a fixed time and free dom enpoints [1]. To it, in particular, reduce approximation problems for function, passenmetered in [1, 2], one of which is the "problem of finding a road profile".

According to its statement, problem (1.1)—(1.3) differes eysentioıly from problems pascnuered, for hanpumle, in works [3—7] in that the lefest kond of the trajectory sdere is not fixed.

For the formulated problem (1.1)—(1.3) it is possible without great diffiiculties to establish the cylectroce and uniqueness of the optimaal trajectorry. In thus, dorase of cyllectroce is carred out by the way, that has already betome traditional, thining in cvetom behaviour of fynktion $F(x, t)$ at beconelnesy u

Figure 1: Figure 1

while the integral with respect to the variable τ is understood as the limit of the corresponding Riemann integral over the interval as $T \to \infty$.

This theorem allows us to obtain the inversion of the integral transform

$$F(\tau) = L[x(t)] = \int_1^\infty x(t) P_{-\frac{1}{2}+i\tau}(t) dt$$

,.() . ( ) ! $f(x)dx$,

$0 < \tau < \infty$

Under the assumption that $f(x)$ belongs to the class of functions satisfying conditions (2), the inversion formula is given by:

$$f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(s)}{\Gamma(1-s)} x^{-s} F(s) ds \quad (5)$$

holding for almost all x. The application of the expansion (3) to functions of a particular form leads to relations that are of interest in the theory of special functions and in mathematical physics.

1. ASYMPTOTIC REPRESENTATIONS OF SPHERICAL FUNCTIONS

The proof of the theorem relies on certain asymptotic representations of Legendre functions for large absolute values of the index. To derive the first of the required representations, we utilize the well-known equality:

$\Gamma(r - \nu)$

(c h a) =

$a > 0, \mathrm{Re}\,\nu > -2$

and introduce a new integration variable $\theta - \alpha$. We then obtain

$$e \approx \omega(\Gamma)\rho^{-\nu} Q, \quad \cosh\alpha = -\frac{\Lambda - \gamma}{\sqrt{2\sinh\alpha}} \int \sqrt{\sinh\theta}\, d\theta$$

G–V*. ” T I E « > •

where $g(t) = 1 - |1 + \coth\alpha - \tanh\gamma|$

The integral i_R can be expressed in terms of gamma functions.

According to Lebedev, it follows that $i = (-\pi n_0 Q \sqrt{1 - |v|^{-2}} \arg v)$. Regarding the integral I_L, by integrating by parts, we find:

$$\frac{g'(t)}{g(t)\cosh t / \sinh^{3/2} t}$$

From this, taking into account that $g(t)$ is a positive, monotonically increasing function, we obtain $\frac{g'(t)}{g(t)\cosh t}$.

$|1| <$

Substituting $g'(t)$ and evaluating the integral on the right-hand side of the final equation using the substitution $u = \coth(y)$, we find:

$$\sqrt{\coth(a)}$$

$v|J(1 + \dots)^{3/2} < \dots \text{cth } a$

Consequently,

$£_1 = \int \text{cth } aO(|v|^{-1}),$

$|v| \to \infty, |\arg v| <$

From equations (7)-(9), we derive the asymptotic representation:

$$\Phi(\cosh\alpha) = \left(e^{-\alpha} + \sqrt{\coth\alpha}\right)$$

$|v| \to \infty, |\arg v| < \dots, a > 0,$

where the symbol depends on a. Deriving an analogous formula for the Legendre function of the second kind is more complex. We proceed from the integral representation:

$d\theta =$

$|\sqrt{2\cosh 2\cosh\theta}$

Assuming $t = a$ and $\mathrm{Re}(\nu) > 0$, we can represent the first of the integrals as a sum:

$$\int_0^{2a} \left(\int_0^t \sinh\tau \, d\tau + \int_t^{2a} \sinh\tau \, d\tau \right)^{-1/2}$$

where $h(t) = (1 - \coth a \cdot \tanh \frac{t}{2})^{-1}$.

1. We have

I apologize, but the provided text consists only of fragmented characters and symbols (“ - ^ (-)”). This input does not contain any coherent academic content, scientific data, or translatable language.

If you have a specific academic manuscript or technical text you would like translated, please provide the full content, and I will be happy to translate it into fluent English while preserving all mathematical notation, citations, and figure markers as requested.

The provided text fragment appears to be a mathematical expression containing variables and a inequality:

$|v| \to \infty, |\arg v| < y$

Next, by integrating by parts, we find:

$$...$$

from which, by estimating the modulus, we obtain:

$$-f^L - = -^U - O(|v|-)$$

$|v| \to \infty, |\arg v| < ....$

By performing integration by parts on the last integral in (12), we can represent it in the following form:

$h(t)dt = \int \cdots = \ldots e \ldots$

$\sqrt{\sinh a} \ldots h'(t)dth(t) \cosh tdt$

Mathematical Analysis

If we observe that $h(t)$ is a positive, monotonically increasing function on the interval $(0, a)$, then the following estimate can be derived from the preceding equality after straightforward transformations:

$$\frac{h'(t)}{h(t)} \leq \frac{s}{t} \frac{h'(t)}{h(t)}$$

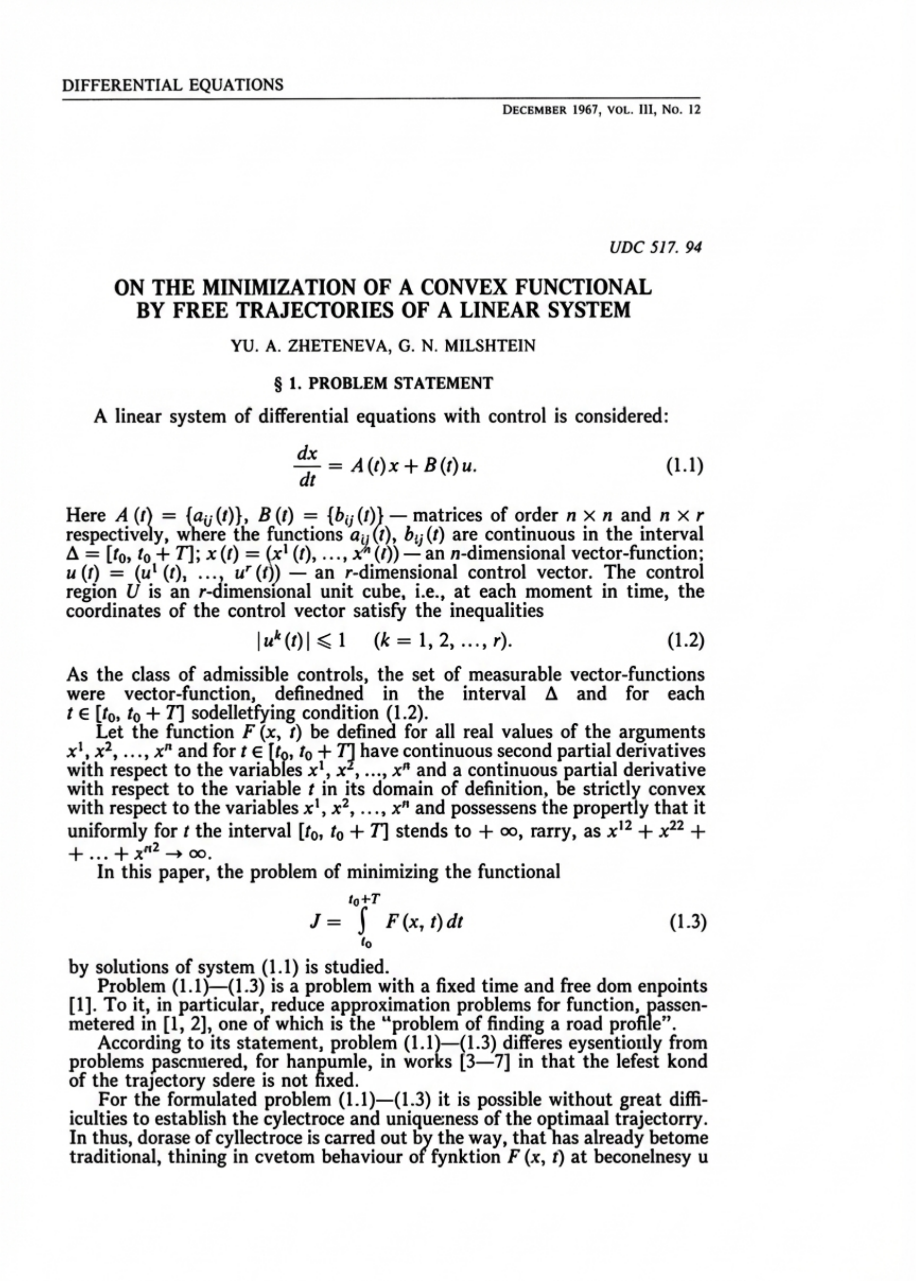
This relationship characterizes the growth rate of the function within the specified domain. By analyzing the differential properties of $h(t)$, we can establish bounds that are critical for the subsequent convergence proofs. The monotonicity ensures that the derivative $h'(t)$ remains non-negative, which simplifies the application of the mean value theorem in this context.

Furthermore, the behavior of the ratio $\frac{h'(t)}{h(t)}$ provides insight into the logarithmic growth of the system. Such estimates are standard in the study of integral transforms and special functions, particularly when determining the asymptotic behavior of solutions near the boundary of the interval $(0, a)$.

$= dt =$

$\sqrt{\cosh a \tanh a}/(\cosh a - \ldots)^{3/2}$, from which it follows, as shown above, that

$$h(t)dt = \sqrt{\cosh a}/\cosh a + O(\nu^{-1})$$



DIFFERENTIAL EQUATIONS

DECEMBER 1967, VOL. III, NO. 12

UDC 517. 94

ON THE MINIMIZATION OF A CONVEX FUNCTIONAL BY FREE TRAJECTORIES OF A LINEAR SYSTEM

YU. A. ZHETENEVA, G. N. MILSHTEIN

§ 1. PROBLEM STATEMENT

A linear system of differential equations with control is considered:

$$\frac{dx}{dt} = A(t)x + B(t)u. \tag{1.1}$$

Here $A(t) = \{a_{ij}(t)\}$, $B(t) = \{b_{ij}(t)\}$ — matrices of order $n \times n$ and $n \times r$ respectively, where the functions $a_{ij}(t)$, $b_{ij}(t)$ are continuous in the interval $\Delta = [t_0, t_0 + T]$; $x(t) = (x^1(t), \ldots, x^n(t))$ — an n-dimensional vector-function; $u(t) = (u^1(t), \ldots, u^r(t))$ — an r-dimensional control vector. The control region U is an r-dimensional unit cube, i.e., at each moment in time, the coordinates of the control vector satisfy the inequalities

$$|u^k(t)| \leqslant 1 \qquad (k = 1, 2, \ldots, r). \tag{1.2}$$

As the class of admissible controls, the set of measurable vector-functions were vector-function, definedned in the interval Δ and for each $t \in [t_0, t_0 + T]$ sodelletfying condition (1.2).

Let the function $F(x, t)$ be defined for all real values of the arguments $x^1, x^2, \ldots, x^n$ and for $t \in [t_0, t_0 + T]$ have continuous second partial derivatives with respect to the variables $x^1, x^2, \ldots, x^n$ and a continuous partial derivative with respect to the variable t in its domain of definition, be strictly convex with respect to the variables $x^1, x^2, \ldots, x^n$ and possessens the propertly that it uniformly for t the interval $[t_0, t_0 + T]$ stends to $+\infty$, rarry, as $x^{12} + x^{22} + \ldots + x^{n2} \to \infty$.

In this paper, the problem of minimizing the functional

$$J = \int_{t_0}^{t_0+T} F(x, t)\, dt \tag{1.3}$$

by solutions of system (1.1) is studied.

Problem (1.1)—(1.3) is a problem with a fixed time and free dom enpoints [1]. To it, in particular, reduce approximation problems for function, passenmetered in [1, 2], one of which is the "problem of finding a road profile".

According to its statement, problem (1.1)—(1.3) differes eysentiouly from problems pascnuered, for hanpumle, in works [3—7] in that the lefest kond of the trajectory sdere is not fixed.

For the formulated problem (1.1)—(1.3) it is possible without great diffiiculties to establish the cylectroce and uniqueness of the optimaal trajectorry. In thus, dorase of cyllectroce is carred out by the way, that has already betome traditional, thining in cvetom behaviour of fynktion $F(x, t)$ at beconelnesy u

Figure 2: Figure 1

(15)

$|v| \to \infty, |\arg v| < y$

It follows from (12)–(15) that

$15, =$

$e \ldots \{1 - f/ \coth a \ldots / \cosh a O(|v|^{-1/2})\}$ $|v| \to \infty, |\arg v| <$

To obtain the asymptotic representation of the integral, we observe that it can be transformed into the following form:

$$\int \frac{d\theta}{2\cosh\theta - 2\cosh\alpha}$$

*) Note: The specific limits and coefficients of this transformation depend on the boundary conditions established in the previous section.

$6 \cdots = d\theta + o \int /2\cosh a + 2\cosh\theta$

where the upper or lower sign is chosen depending on the sign of Im $\nu \gtrless 0$.

$ni \int j \ldots 2\cosh a - 2\cos\psi$

Formula (17) for Im $\nu < 0$ is obtained by integrating the function $e^{\nu\alpha}(2\cosh\alpha - 2\cosh\theta)^{-1/2}$ along a contour formed by the semi-infinite line $(\theta, \theta + i\infty)$, the segment $[-\theta, \theta]$, and the semi-infinite line $(-\theta, -\theta + i\infty)$, while bypassing the branch points. For Im $\nu > 0$, the chosen contour is the mirror image of the previous contour relative to the real axis.

The asymptotic representation of the first integral in (17) is given by formula (10). By performing integration by parts and estimating the resulting integrals in terms of their moduli, we obtain the asymptotic representations for the remaining two integrals: $O(|\nu|^{-1/2} e^{\mathrm{Re}\ \nu\alpha + 2\cosh\theta})$.

$|\arg\nu| < \frac{\pi}{2}$

$\sqrt{2\cosh a - 2\cos \ldots}$

$\infty, |\arg v| < \gamma$

From (17)-(19) it follows that $\frac{z\nu}{\sinh a} / \frac{\nu}{\sinh a}$

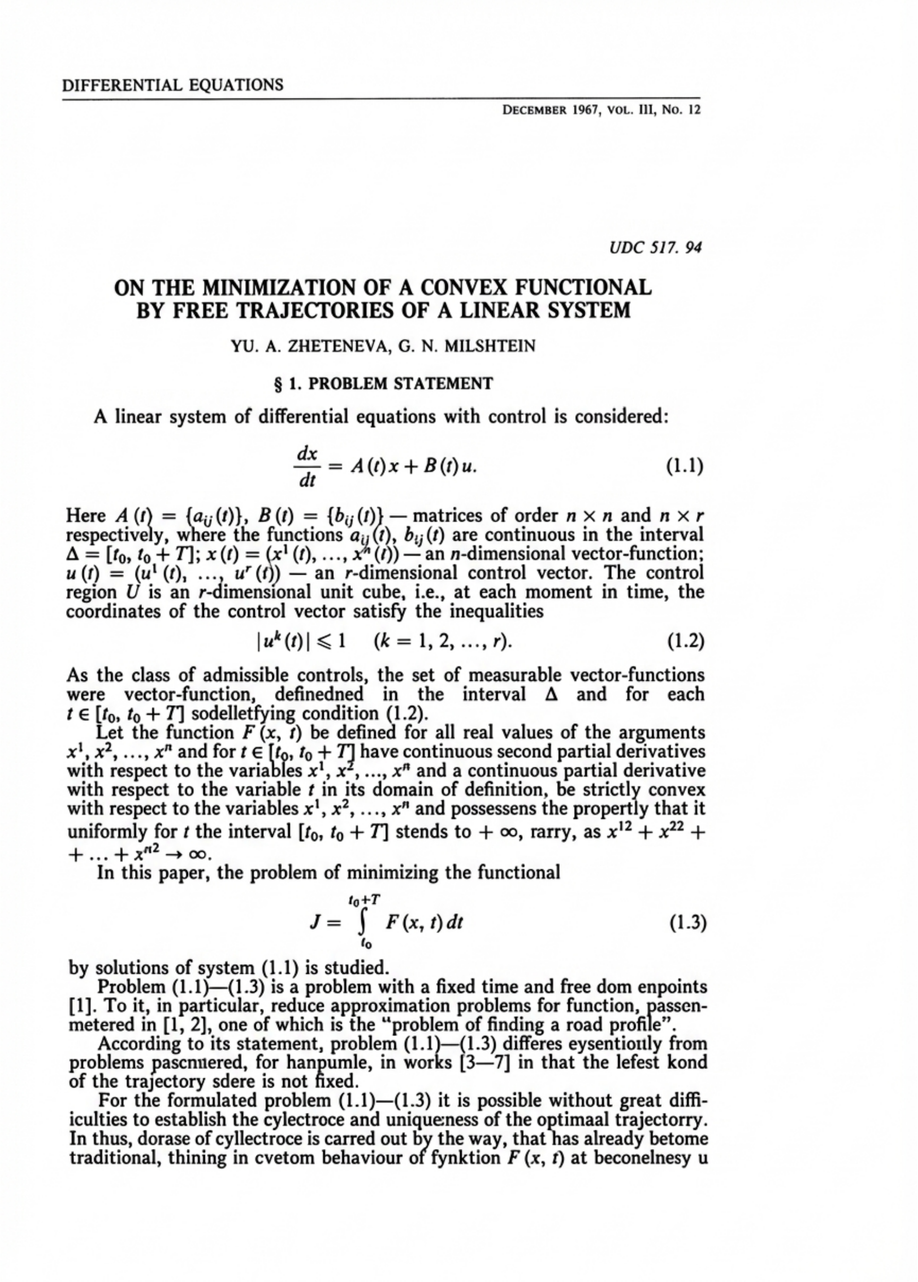
$|v| \to \infty, |\arg v| < \gamma.$

Formulas (11), (16), and (20) show that the following asymptotic representation holds:

$\left(\frac{1}{\pi}\right)^{1/2} P_1(\cosh a) = \ldots$

$\sqrt{2\pi\nu\sinh a}/(ie \cdots + e/\coth a/\cosh a O(|v|^{-1}))$

$|v| \to \infty, \quad |\arg v| < \gamma, \quad a > 0,$



DIFFERENTIAL EQUATIONS

DECEMBER 1967, VOL. III, NO. 12

UDC 517. 94

ON THE MINIMIZATION OF A CONVEX FUNCTIONAL BY FREE TRAJECTORIES OF A LINEAR SYSTEM

YU. A. ZHETENEVA, G. N. MILSHTEIN

§ 1. PROBLEM STATEMENT

A linear system of differential equations with control is considered:

$$\frac{dx}{dt} = A(t)x + B(t)u. \tag{1.1}$$

Here $A(t) = \{a_{ij}(t)\}$, $B(t) = \{b_{ij}(t)\}$ — matrices of order $n \times n$ and $n \times r$ respectively, where the functions $a_{ij}(t)$, $b_{ij}(t)$ are continuous in the interval $\Delta = [t_0, t_0 + T]$; $x(t) = (x^1(t), \ldots, x^n(t))$ — an n-dimensional vector-function; $u(t) = (u^1(t), \ldots, u^r(t))$ — an r-dimensional control vector. The control region U is an r-dimensional unit cube, i.e., at each moment in time, the coordinates of the control vector satisfy the inequalities

$$|u^k(t)| \leqslant 1 \quad (k = 1, 2, \ldots, r). \tag{1.2}$$

As the class of admissible controls, the set of measurable vector-functions were vector-function, definedned in the interval Δ and for each $t \in [t_0, t_0 + T]$ sodelletfying condition (1.2).

Let the function $F(x, t)$ be defined for all real values of the arguments $x^1, x^2, \ldots, x^n$ and for $t \in [t_0, t_0 + T]$ have continuous second partial derivatives with respect to the variables $x^1, x^2, \ldots, x^n$ and a continuous partial derivative with respect to the variable t in its domain of definition, be strictly convex with respect to the variables $x^1, x^2, \ldots, x^n$ and possessens the propertly that it uniformly for t the interval $[t_0, t_0 + T]$ stends to $+\infty$, rarry, as $x^{12} + x^{22} + \ldots + x^{n2} \to \infty$.

In this paper, the problem of minimizing the functional

$$J = \int_{t_0}^{t_0+T} F(x, t)\,dt \tag{1.3}$$

by solutions of system (1.1) is studied.

Problem (1.1)—(1.3) is a problem with a fixed time and free dom enpoints [1]. To it, in particular, reduce approximation problems for function, passenmetered in [1, 2], one of which is the “problem of finding a road profile”.

According to its statement, problem (1.1)—(1.3) differes eysentiouly from problems pascnuered, for hanpumle, in works [3—7] in that the lefest kond of the trajectory sdere is not fixed.

For the formulated problem (1.1)—(1.3) it is possible without great diffiiculties to establish the cylectroce and uniqueness of the optimaal trajectorry. In thus, dorase of cyllectroce is carred out by the way, that has already betome traditional, thining in cvetom behaviour of fynktion $F(x, t)$ at beconelnesy u

Figure 3: Figure 1

where the sign ± is chosen accordingly, and the symbol depends on a. The derived asymptotic formulas (10) and (21) differ from conventional asymptotic representations in that they provide an estimate of the remainder terms that remains valid across the entire interval of the variable' s variation.

§ 2. PROOF OF THE DECOMPOSITION THEOREM

Proceeding to the proof of equality (3), we first note that from the integral representation for the product of spherical functions:

LEBEDEV

$1 < x < \infty, \tau > 0,$

and the inequality $|(\cdot)| \leq ...$, the following estimate follows:

$< 2P_{-\frac{1}{2}+i\tau}(x)Q_{-\frac{1}{2}+i\tau}(x),$

$1 < x < \infty, \tau > 0.$

It follows that the inner integral on the right-hand side of the formula is majorized by the integral $\int_1^\infty |Q_\nu(y)||f(y)|dy$, which converges due to the conditions imposed on the function $f(x)$. Furthermore, this integral represents a continuous function on the interval, and the iterated integral is well-defined.

$$J(x,T) = \frac{1}{2}\int_1^x \frac{1}{\sqrt{x^2-1}} \left[\int_0^T \tanh(\pi\tau)P_{-\frac{1}{2}+i\tau}(x)d\tau\right] dx$$

The expression $\int[Q_\nu(y) + Q_{-\nu}(y)]f(y)dy$ is meaningful. Furthermore, in view of dominated convergence, the order of integration can be interchanged, such that:

$$\int f(y)G(x,y,T)dy = \int \left[\int (Q_\nu(y) + Q_{-\nu}(y))f(y)dy\right] dT$$

$1 < x < \infty, \quad 1 < y < \infty, \quad T > 0.$

By setting these values and utilizing well-known functional relations, we obtain:

$P_{-\nu-1} \cdots = P \ldots$

Note that $\Gamma \approx 0\left(\frac{1}{l}\ln(w-1)\right)$.

$-2 \approx y \approx O(\ln\eta)$, where $a < y < \infty$.

By substituting $P(x) = Q_{-\frac{1}{2}}(x) - Q_{\frac{1}{2}}(x)$, we obtain the integral representation:

$G(x,y,T) = \sqrt{T}\sqrt{y}\sqrt{T-x}$

The expression found is convenient for studying the function when $y > x$. To achieve this, we rewrite (28) in the form:

$G(x,y,T) = \sqrt{T-x}\sqrt{y} - \sqrt{y-x}$

$WQ_W P(y)Q(t/)dv$. Taking into account that the integrand in the first integral is an odd function, it follows that this integral is equal to zero. Thus, we obtain:

$G(x, y, T) = \sqrt{y^i}$

$(y)dv$. The functions under the integral signs in (28) and (29) possess no singularities in the half-plane. Consequently, the integration along the segment of the imaginary axis can be replaced by integration along a circular arc of radius lying within this half-plane. Thus, we have:

$G(x, y, T) = \sqrt{x^2 - T}\sqrt{T^2 - |y - x|^2}$

$v \dots J \dots) f[Q \dots]^2 \dots$ (30)

$1 < x < y$.

N. N. LEBEDEV $G(x, y, \tau) = 0$. By subsequently setting $x = \cosh\alpha$ and $y = \cosh\alpha'$, we can write formula (25) in the form:

$\Gamma = \int f(\cosh\alpha') G(\cosh\alpha, \cosh\alpha', \tau) \sinh\alpha' \, d\alpha' =$

$+ \int f(\cosh\alpha') G(\cosh\alpha, \cosh\alpha', T) \sinh\alpha' d\alpha' =$

$$I = I_1(\lambda, \Gamma) + I_2(\lambda, \Gamma). \tag{32}$$

To calculate the limit of the integral as $\lambda \to \infty$, we utilize formula (31) and the asymptotic representations given in (21). On the arc Γ_2,

$-\frac{\pi}{2} < \phi < \frac{\pi}{2}$; therefore,

$(\cosh a) Q_i(\cosh a') Q_i(\cosh a') I - 2v(a4 - a') - 2v(a - a') 4 \sinh a \sinh a' I \dots$

Substituting (33) into the first integral of (31), we obtain:

$$\int_x^1 P_\nu(y)\, dy = \frac{1 - x^2}{2\nu + 1}\left[P_{\nu-1}(x) - P_{\nu+1}(x)\right]$$

$= 1 +$

We have $-2va'$, from which it follows:

$$-2va - 2v(a + a')/\cosh(T)$$

$2va' f0(1), 0 < a' < 5, dv = \dots$

$V \cdots < a' < a, \dots$

depends similarly

$dv = 0(7, -x)$,

$1 \dots e^{-2v(a+a')}$

$dv = 0(T^{-1})$

for all $0 < a'$. From this, we conclude that

$$\int XP_i(y)Q_1(y)dy =$$

LEBEDEV $1 + o(1)$

$0 < a' \leq b$

$1 + 0$

$+ \coth a' \sqrt{c} W_0(T - 1/2)$, where $0 < a' < a$.

Thus, based on $V(V - \dots) + \coth(1/2)(r - 1/2) < a' < a$, it follows from (36) that

$G(\cosh a, \cosh a', T) =$

I apologize, but the provided text appears to be a fragmented collection of mathematical symbols and Cyrillic characters (e.g., “cth” , “ ” , “ ”) that do not form coherent academic sentences or a structured scientific argument.

If this is a specific mathematical derivation or a fragment from a technical paper, please provide the full context or the complete paragraph. To ensure an accurate and readable academic translation, I require sufficient text to establish the logical flow and technical meaning of the expressions.

$1 + 0$

$+ \coth a' \cosh a' \cdots - 1/2), 6 < a' < a - \dots$

By partitioning the integration interval into the sub-intervals $(0, \delta)$, $(\delta, a - \delta)$, and $(a - \delta, a)$, and applying equation (38), we obtain:

$$I = J_x(x, T) = \int f(\cosh a') \sinh a' \, da' - f_0$$

$+0(1) \int |f(\cosh a')| \sinh a' da' + 0(1) \int |f(\cosh a')| \sinh a' da' \dots$

$x \cosh SJ, (x, T) = \int f(y)dy + 0(1) \int |f(y)|dy+$

$\int |f(y)|dy + o \dots \int |f(y)|dy$

I apologize, but the provided text consists of fragmented characters and symbols that do not form coherent academic content or a translatable scientific text.

If you have a specific academic manuscript or a section of a paper you would like translated, please provide the full text, and I will be ensure it is translated into fluent, academic English while preserving all mathematical notation, citations, and figure markers as requested.

Taking γ to be sufficiently small and then increasing T, we can make all integrals (with the exception of the first) arbitrarily small. Therefore, by an analogous process based on equations (30) and (34), and under the assumption that the second condition of (32) is satisfied, the theorem is proven. Thus, we have established the following result.

Im $J(x, \Gamma) = \int \int f(y) dy$, (42)

This concludes the proof. In conclusion, it should be noted that the conditions imposed on the function $f(x)$ are sufficient; however, the theorem may hold under less restrictive assumptions. In particular, the theorem remains valid for functions satisfying the following conditions:

$f(x) - y = m'$

$f(x) \ln x \in L^a$, where $a > 1$ is a certain constant. The validity of this assertion regarding the function can be verified directly, after which the possibility of extending Theorem (3) to this class of functions becomes evident.

§ 3. EXAMPLES OF EXPANDING FUNCTIONS INTO INTEGRALS

On Squares of Legendre Functions: We present several interesting examples of expansions of this type, which can be derived from the general formula through an appropriate selection of the function.

$\frac{1}{x} \int_0^\infty \tau \tanh \pi\tau P_{-\frac{1}{2}+i\tau}(x) d\tau = \frac{1}{x}$

Here, the first of the conditions (2) is essential.

LEBEDEV $\frac{x+1}{2}$

$\int_0^\infty \tau \tanh \pi\tau P_{-\frac{1}{2}+i\tau}(x) \Gamma\left(\frac{1}{4} + \frac{i\tau}{2}\right) \Gamma\left(\frac{1}{4} - \frac{i\tau}{2}\right) d\tau = \frac{\sqrt{2}\pi}{\sqrt{x^2-1}}$

(46) $\frac{1}{\sqrt{2x}}$

$a > -1,$

$\arcsin \sqrt{1-a}$

$-1 < a < 1,$

$-$ + Va + r t ~ T + ’ «

$a > 1,$

$\int_X \Gamma(v - IT)[P^*] dx,$

1 Re

In these formulas, the function on the left side of the equality is related to the function $\phi(x)$ by the relation $f(x) = [\phi(x)]'$. The variable x is defined such that the resulting equalities remain valid even at $x = 1$.

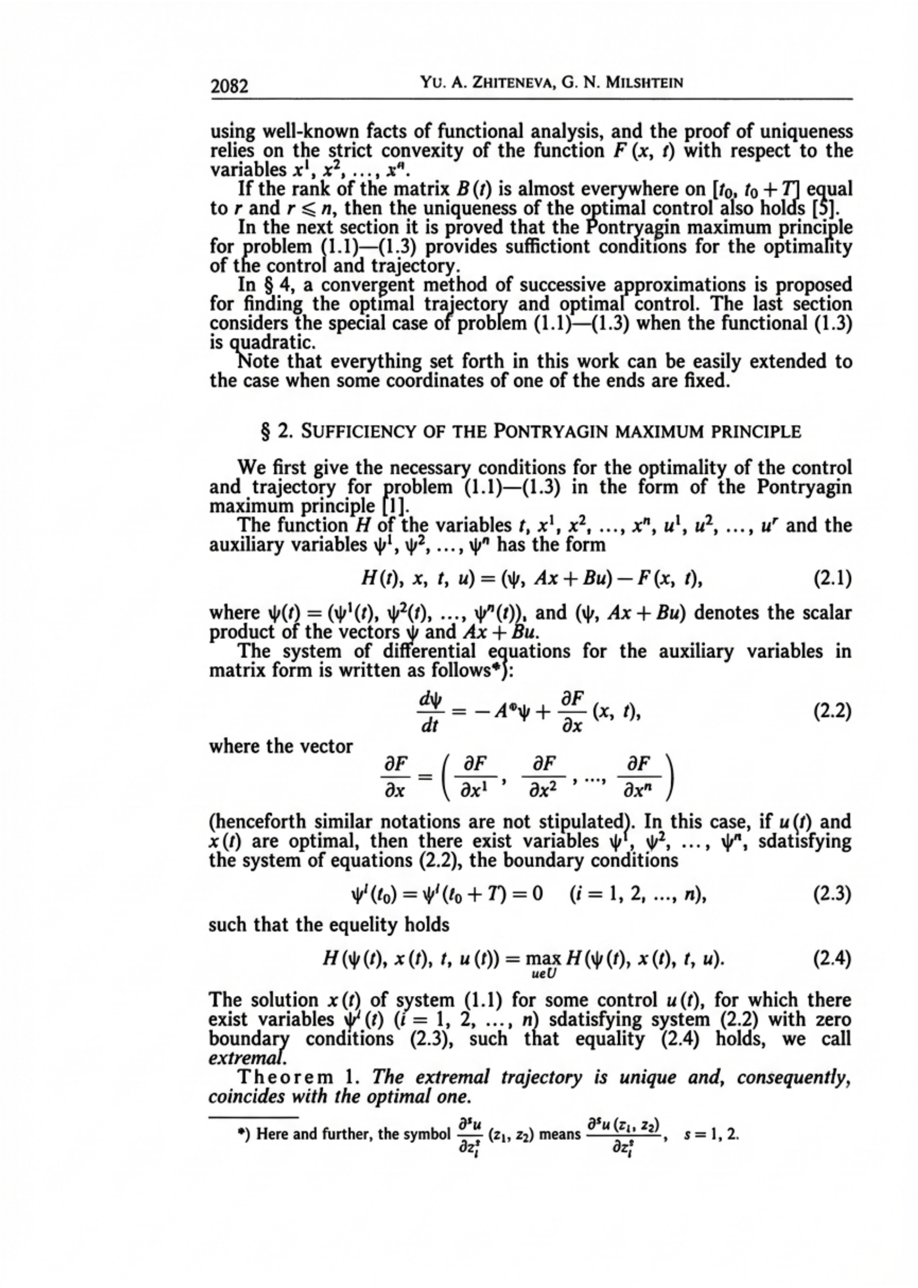
The values of the function $\Phi(m)$, defined by equality (4), are determined by replacing the product of spherical functions with an integral and utilizing the Mehler–Fock transform tables [**?**, **?**].

References

Mehler, F. G. "Ueber eine Kugel- und Cylinder-Funktionen verwandte Funktion und ihre Anwendung in der Theorie der Elektricitäts-verteilung." *Mathematische Annalen*, Vol. 18. Fock, V. A. *Doklady Akademii Nauk SSSR*, 279-282. Lebedev, N. N. *Special Functions and Their Applications.* Moscow, 1963.

Tables of Lebedev, Mehler, and Generalized Mehler Transforms. Boeing Scientific Research Laboratories, *Tables of Integral Transforms*, Vol. II, 1954. Ioffe Physico-Technical Institute. Received July 14, 1966.

Figures



using well-known facts of functional analysis, and the proof of uniqueness relies on the strict convexity of the function $F(x, t)$ with respect to the variables $x^1, x^2, \ldots, x^n$.

If the rank of the matrix $B(t)$ is almost everywhere on $[t_0, t_0+T]$ equal to r and $r \leqslant n$, then the uniqueness of the optimal control also holds [5].

In the next section it is proved that the Pontryagin maximum principle for problem (1.1)—(1.3) provides suffictiont conditions for the optimality of the control and trajectory.

In § 4, a convergent method of successive approximations is proposed for finding the optimal trajectory and optimal control. The last section considers the special case of problem (1.1)—(1.3) when the functional (1.3) is quadratic.

Note that everything set forth in this work can be easily extended to the case when some coordinates of one of the ends are fixed.

§ 2. SUFFICIENCY OF THE PONTRYAGIN MAXIMUM PRINCIPLE

We first give the necessary conditions for the optimality of the control and trajectory for problem (1.1)—(1.3) in the form of the Pontryagin maximum principle [1].

The function H of the variables $t, x^1, x^2, \ldots, x^n, u^1, u^2, \ldots, u^r$ and the auxiliary variables $\psi^1, \psi^2, \ldots, \psi^n$ has the form

$$H(t), x, t, u) = (\psi, Ax + Bu) - F(x, t), \tag{2.1}$$

where $\psi(t) = (\psi^1(t), \psi^2(t), \ldots, \psi^n(t))$, and $(\psi, Ax + Bu)$ denotes the scalar product of the vectors ψ and $Ax + Bu$.

The system of differential equations for the auxiliary variables in matrix form is written as follows*):

$$\frac{d\psi}{dt} = -A^*\psi + \frac{\partial F}{\partial x}(x, t), \tag{2.2}$$

where the vector

$$\frac{\partial F}{\partial x} = \left(\frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^2}, \ldots, \frac{\partial F}{\partial x^n}\right)$$

(henceforth similar notations are not stipulated). In this case, if $u(t)$ and $x(t)$ are optimal, then there exist variables $\psi^1, \psi^2, \ldots, \psi^n$, sdatisfying the system of equations (2.2), the boundary conditions

$$\psi^i(t_0) = \psi^i(t_0 + T) = 0 \quad (i = 1, 2, \ldots, n), \tag{2.3}$$

such that the equelity holds

$$H(\psi(t), x(t), t, u(t)) = \max_{u \in U} H(\psi(t), x(t), t, u). \tag{2.4}$$

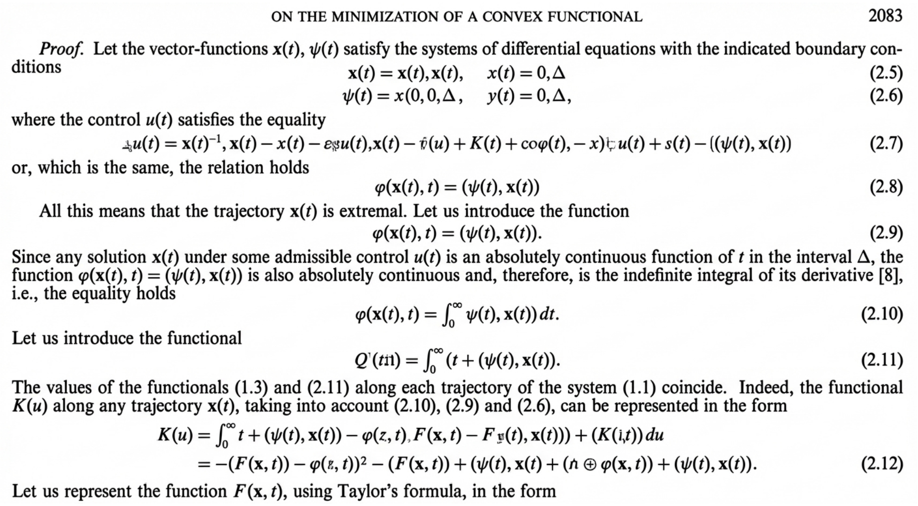
The solution $x(t)$ of system (1.1) for some control $u(t)$, for which there exist variables $\psi^i(t)$ $(i = 1, 2, \ldots, n)$ sdatisfying system (2.2) with zero boundary conditions (2.3), such that equality (2.4) holds, we call *extremal*.

Theorem 1. *The extremal trajectory is unique and, consequently, coincides with the optimal one.*

---

*) Here and further, the symbol $\frac{\partial^s u}{\partial z_i^s}(z_1, z_2)$ means $\frac{\partial^s u(z_1, z_2)}{\partial z_i^s}$, $s = 1, 2$.

Figure 4: Figure 2



ON THE MINIMIZATION OF A CONVEX FUNCTIONAL 2083

Proof. Let the vector-functions $\mathbf{x}(t)$, $\psi(t)$ satisfy the systems of differential equations with the indicated boundary conditions

$$\mathbf{x}(t) = \mathbf{x}(t), \mathbf{x}(t), \quad x(t) = 0, \Delta \tag{2.5}$$

$$\psi(t) = x(0, 0, \Delta, \quad y(t) = 0, \Delta, \tag{2.6}$$

where the control $u(t)$ satisfies the equality

$$u(t) = \mathbf{x}(t)^{-1}, \mathbf{x}(t) - x(t) - \varepsilon u(t), \mathbf{x}(t) - \hat{v}(u) + K(t) + \mathrm{co}\varphi(t), - x) u(t) + s(t) - ((\psi(t), \mathbf{x}(t)) \tag{2.7}$$

or, which is the same, the relation holds

$$\varphi(\mathbf{x}(t), t) = (\psi(t), \mathbf{x}(t)) \tag{2.8}$$

All this means that the trajectory $\mathbf{x}(t)$ is extremal. Let us introduce the function

$$\varphi(\mathbf{x}(t), t) = (\psi(t), \mathbf{x}(t)). \tag{2.9}$$

Since any solution $\mathbf{x}(t)$ under some admissible control $u(t)$ is an absolutely continuous function of t in the interval Δ, the function $\varphi(\mathbf{x}(t), t) = (\psi(t), \mathbf{x}(t))$ is also absolutely continuous and, therefore, is the indefinite integral of its derivative [8], i.e., the equality holds

$$\varphi(\mathbf{x}(t), t) = \int_0^\infty \psi(t), \mathbf{x}(t))\, dt. \tag{2.10}$$

Let us introduce the functional

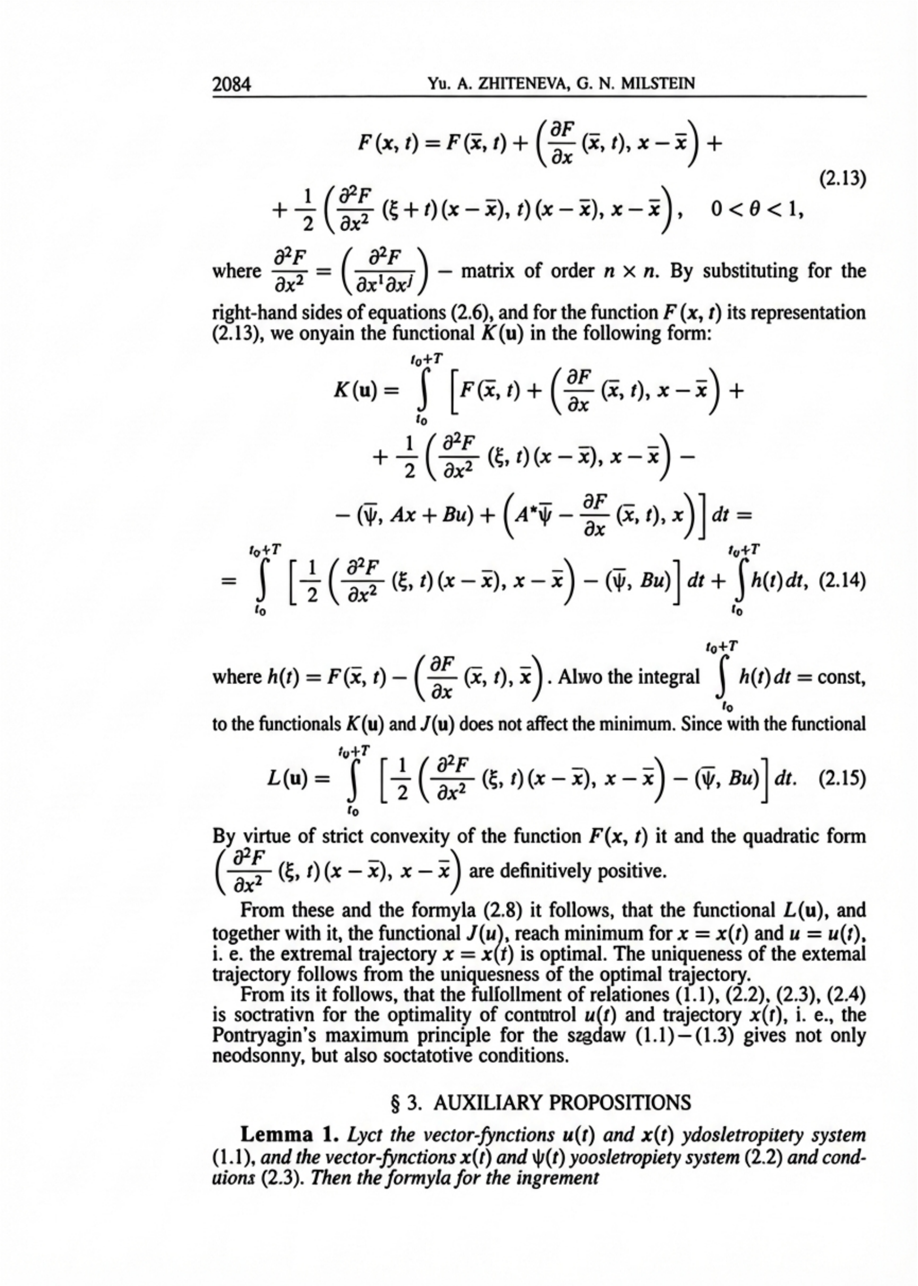
$$Q(t1) = \int_0^\infty (t + (\psi(t), \mathbf{x}(t)). \tag{2.11}$$

The values of the functionals (1.3) and (2.11) along each trajectory of the system (1.1) coincide. Indeed, the functional $K(u)$ along any trajectory $\mathbf{x}(t)$, taking into account (2.10), (2.9) and (2.6), can be represented in the form

$$\begin{aligned} K(u) &= \int_0^\infty t + (\psi(t), \mathbf{x}(t)) - \varphi(z, t), F(\mathbf{x}, t) - F(t), \mathbf{x}(t))) + (K(1, t))\, du \\ &= -(F(\mathbf{x}, t)) - \varphi(z, t))^2 - (F(\mathbf{x}, t)) + (\psi(t), \mathbf{x}(t) + (n \oplus \varphi(\mathbf{x}, t)) + (\psi(t), \mathbf{x}(t)). \end{aligned} \tag{2.12}$$

Let us represent the function $F(\mathbf{x}, t)$, using Taylor's formula, in the form

Figure 5: Figure 3



$$F(x, t) = F(\bar{x}, t) + \left(\frac{\partial F}{\partial x}(\bar{x}, t), x - \bar{x}\right) + \\ + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(\xi + t)(x - \bar{x}), t)(x - \bar{x}), x - \bar{x}\right), \quad 0 < \theta < 1, \tag{2.13}$$

where $\frac{\partial^2 F}{\partial x^2} = \left(\frac{\partial^2 F}{\partial x^1 \partial x^j}\right)$ – matrix of order $n \times n$. By substituting for the right-hand sides of equations (2.6), and for the function $F(x, t)$ its representation (2.13), we onyain the functional $K(\mathbf{u})$ in the following form:

$$K(\mathbf{u}) = \int\limits_{t_0}^{t_0+T} \Big[F(\bar{x}, t) + \left(\frac{\partial F}{\partial x}(\bar{x}, t), x - \bar{x}\right) + \\ + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(\xi, t)(x - \bar{x}), x - \bar{x}\right) - \\ - (\bar{\psi}, Ax + Bu) + \left(A^*\bar{\psi} - \frac{\partial F}{\partial x}(\bar{x}, t), x\right)\Big]\, dt = \\ = \int\limits_{t_0}^{t_0+T} \left[\frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(\xi, t)(x - \bar{x}), x - \bar{x}\right) - (\bar{\psi}, Bu)\right] dt + \int\limits_{t_0}^{t_0+T} h(t)\, dt, \tag{2.14}$$

where $h(t) = F(\bar{x}, t) - \left(\frac{\partial F}{\partial x}(\bar{x}, t), \bar{x}\right)$. Alwo the integral $\int\limits_{t_0}^{t_0+T} h(t)\, dt = \text{const}$, to the functionals $K(\mathbf{u})$ and $J(\mathbf{u})$ does not affect the minimum. Since with the functional

$$L(\mathbf{u}) = \int\limits_{t_0}^{t_0+T} \left[\frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(\xi, t)(x - \bar{x}), x - \bar{x}\right) - (\bar{\psi}, Bu)\right] dt. \tag{2.15}$$

By virtue of strict convexity of the function $F(x, t)$ it and the quadratic form $\left(\frac{\partial^2 F}{\partial x^2}(\xi, t)(x - \bar{x}), x - \bar{x}\right)$ are definitively positive.

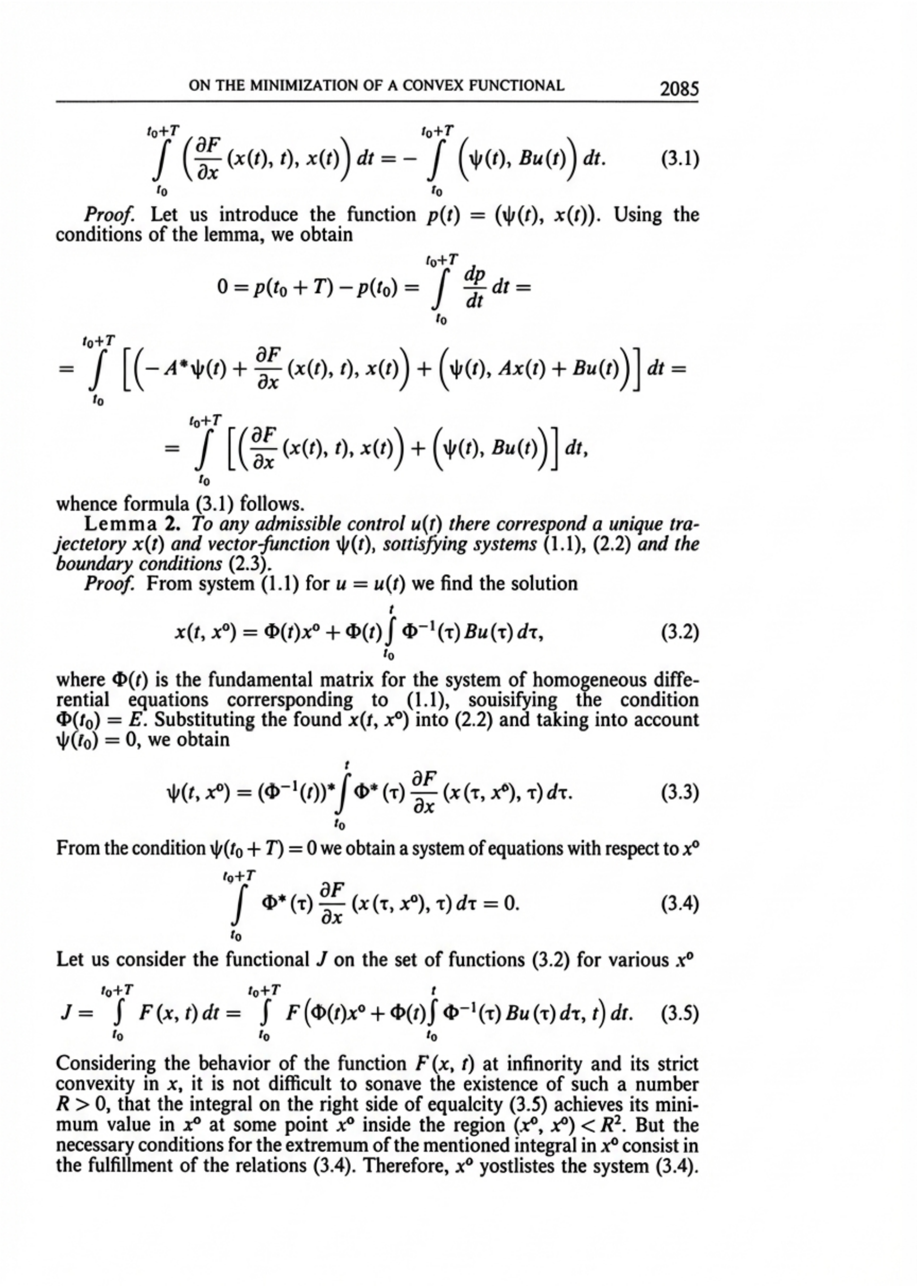
From these and the formyla (2.8) it follows, that the functional $L(\mathbf{u})$, and together with it, the functional $J(u)$, reach minimum for $x = x(t)$ and $u = u(t)$, i. e. the extremal trajectory $x = x(t)$ is optimal. The uniqueness of the extemal trajectory follows from the uniquesness of the optimal trajectory.

From its it follows, that the fulfollment of relationes (1.1), (2.2), (2.3), (2.4) is soctrativn for the optimality of contutrol $u(t)$ and trajectory $x(t)$, i. e., the Pontryagin's maximum principle for the szgdaw (1.1)–(1.3) gives not only neodsonny, but also soctatotive conditions.

§ 3. AUXILIARY PROPOSITIONS

Lemma 1. *Lyct the vector-fynctions* $\mathbf{u}(t)$ *and* $\mathbf{x}(t)$ *ydosletropitety system* (1.1), *and the vector-fynctions* $\mathbf{x}(t)$ *and* $\psi(t)$ *yoosletropiety system* (2.2) *and conduions* (2.3). *Then the formyla for the ingrement*

Figure 6: Figure 4



$$\int\limits_{t_0}^{t_0+T}\left(\frac{\partial F}{\partial x}(x(t),\,t),\,x(t)\right)dt=-\int\limits_{t_0}^{t_0+T}\Big(\psi(t),\,Bu(t)\Big)\,dt. \tag{3.1}$$

Proof. Let us introduce the function $p(t)=(\psi(t),\,x(t))$. Using the conditions of the lemma, we obtain

$$0=p(t_0+T)-p(t_0)=\int\limits_{t_0}^{t_0+T}\frac{dp}{dt}\,dt=$$

$$=\int\limits_{t_0}^{t_0+T}\left[\left(-A^*\psi(t)+\frac{\partial F}{\partial x}(x(t),\,t),\,x(t)\right)+\Big(\psi(t),\,Ax(t)+Bu(t)\Big)\right]dt=$$

$$=\int\limits_{t_0}^{t_0+T}\left[\left(\frac{\partial F}{\partial x}(x(t),\,t),\,x(t)\right)+\Big(\psi(t),\,Bu(t)\Big)\right]dt,$$

whence formula (3.1) follows.

Lemma 2. *To any admissible control $u(t)$ there correspond a unique trajectetory $x(t)$ and vector-function $\psi(t)$, sottisfying systems* (1.1), (2.2) *and the boundary conditions* (2.3).

Proof. From system (1.1) for $u=u(t)$ we find the solution

$$x(t,\,x^{\circ})=\Phi(t)x^{\circ}+\Phi(t)\int\limits_{t_0}^{t}\Phi^{-1}(\tau)\,Bu(\tau)\,d\tau, \tag{3.2}$$

where $\Phi(t)$ is the fundamental matrix for the system of homogeneous differential equations corrersponding to (1.1), souisifying the condition $\Phi(t_0)=E$. Substituting the found $x(t,\,x^{\circ})$ into (2.2) and taking into account $\psi(t_0)=0$, we obtain

$$\psi(t,\,x^{\circ})=(\Phi^{-1}(t))^*\int\limits_{t_0}^{t}\Phi^*(\tau)\,\frac{\partial F}{\partial x}(x(\tau,\,x^{\circ}),\,\tau)\,d\tau. \tag{3.3}$$

From the condition $\psi(t_0+T)=0$ we obtain a system of equations with respect to x°

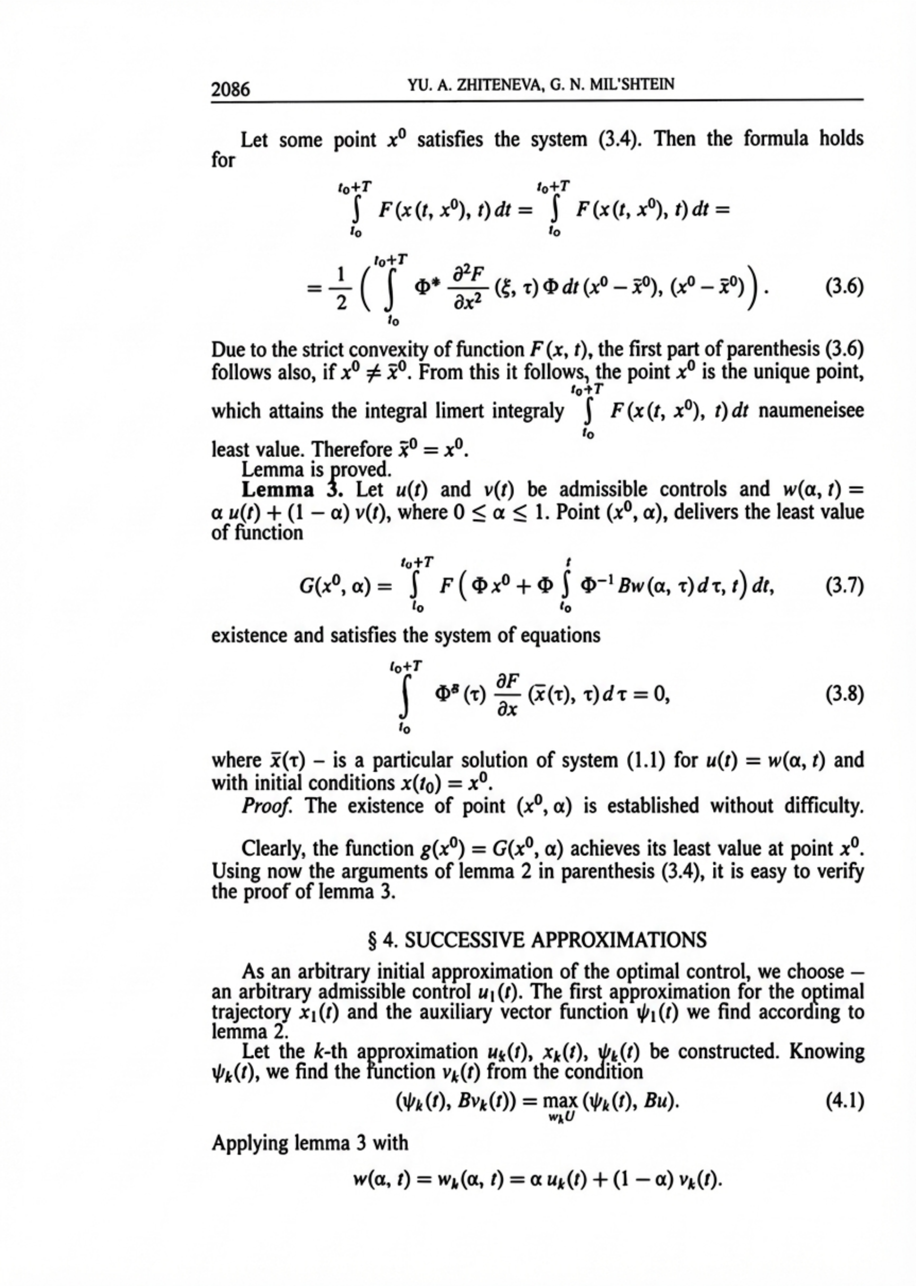
$$\int\limits_{t_0}^{t_0+T}\Phi^*(\tau)\,\frac{\partial F}{\partial x}(x(\tau,\,x^{\circ}),\,\tau)\,d\tau=0. \tag{3.4}$$

Let us consider the functional J on the set of functions (3.2) for various x°

$$J=\int\limits_{t_0}^{t_0+T}F(x,\,t)\,dt=\int\limits_{t_0}^{t_0+T}F\left(\Phi(t)x^{\circ}+\Phi(t)\int\limits_{t_0}^{t}\Phi^{-1}(\tau)\,Bu(\tau)\,d\tau,\,t\right)dt. \tag{3.5}$$

Considering the behavior of the function $F(x,\,t)$ at infinority and its strict convexity in x, it is not difficult to sonave the existence of such a number $R>0$, that the integral on the right side of equalcity (3.5) achieves its minimum value in x° at some point x° inside the region $(x^{\circ},\,x^{\circ})<R^2$. But the necessary conditions for the extremum of the mentioned integral in x° consist in the fulfillment of the relations (3.4). Therefore, x° yostlistes the system (3.4).

Figure 7: Figure 5



Let some point x^0 satisfies the system (3.4). Then the formula holds for

$$\int_{t_0}^{t_0+T} F(x(t, x^0), t)\, dt = \int_{t_0}^{t_0+T} F(x(t, x^0), t)\, dt =$$

$$= \frac{1}{2}\left(\int_{t_0}^{t_0+T} \Phi^* \frac{\partial^2 F}{\partial x^2}(\xi, \tau)\, \Phi\, dt\, (x^0 - \bar{x}^0), (x^0 - \bar{x}^0)\right). \tag{3.6}$$

Due to the strict convexity of function $F(x, t)$, the first part of parenthesis (3.6) follows also, if $x^0 \neq \bar{x}^0$. From this it follows, the point x^0 is the unique point, which attains the integral limert integraly $\int_{t_0}^{t_0+T} F(x(t, x^0), t)\, dt$ naumeneisee least value. Therefore $\bar{x}^0 = x^0$.

Lemma is proved.

Lemma 3. Let $u(t)$ and $v(t)$ be admissible controls and $w(\alpha, t) = \alpha\, u(t) + (1 - \alpha)\, v(t)$, where $0 \leq \alpha \leq 1$. Point (x^0, α), delivers the least value of function

$$G(x^0, \alpha) = \int_{t_0}^{t_0+T} F\left(\Phi x^0 + \Phi \int_{t_0}^{t} \Phi^{-1} B w(\alpha, \tau)\, d\tau, t\right) dt, \tag{3.7}$$

existence and satisfies the system of equations

$$\int_{t_0}^{t_0+T} \Phi^{s}(\tau) \frac{\partial F}{\partial x}(\bar{x}(\tau), \tau)\, d\tau = 0, \tag{3.8}$$

where $\bar{x}(\tau)$ – is a particular solution of system (1.1) for $u(t) = w(\alpha, t)$ and with initial conditions $x(t_0) = x^0$.

Proof. The existence of point (x^0, α) is established without difficulty.

Clearly, the function $g(x^0) = G(x^0, \alpha)$ achieves its least value at point x^0. Using now the arguments of lemma 2 in parenthesis (3.4), it is easy to verify the proof of lemma 3.

§ 4. SUCCESSIVE APPROXIMATIONS

As an arbitrary initial approximation of the optimal control, we choose – an arbitrary admissible control $u_1(t)$. The first approximation for the optimal trajectory $x_1(t)$ and the auxiliary vector function $\psi_1(t)$ we find according to lemma 2.

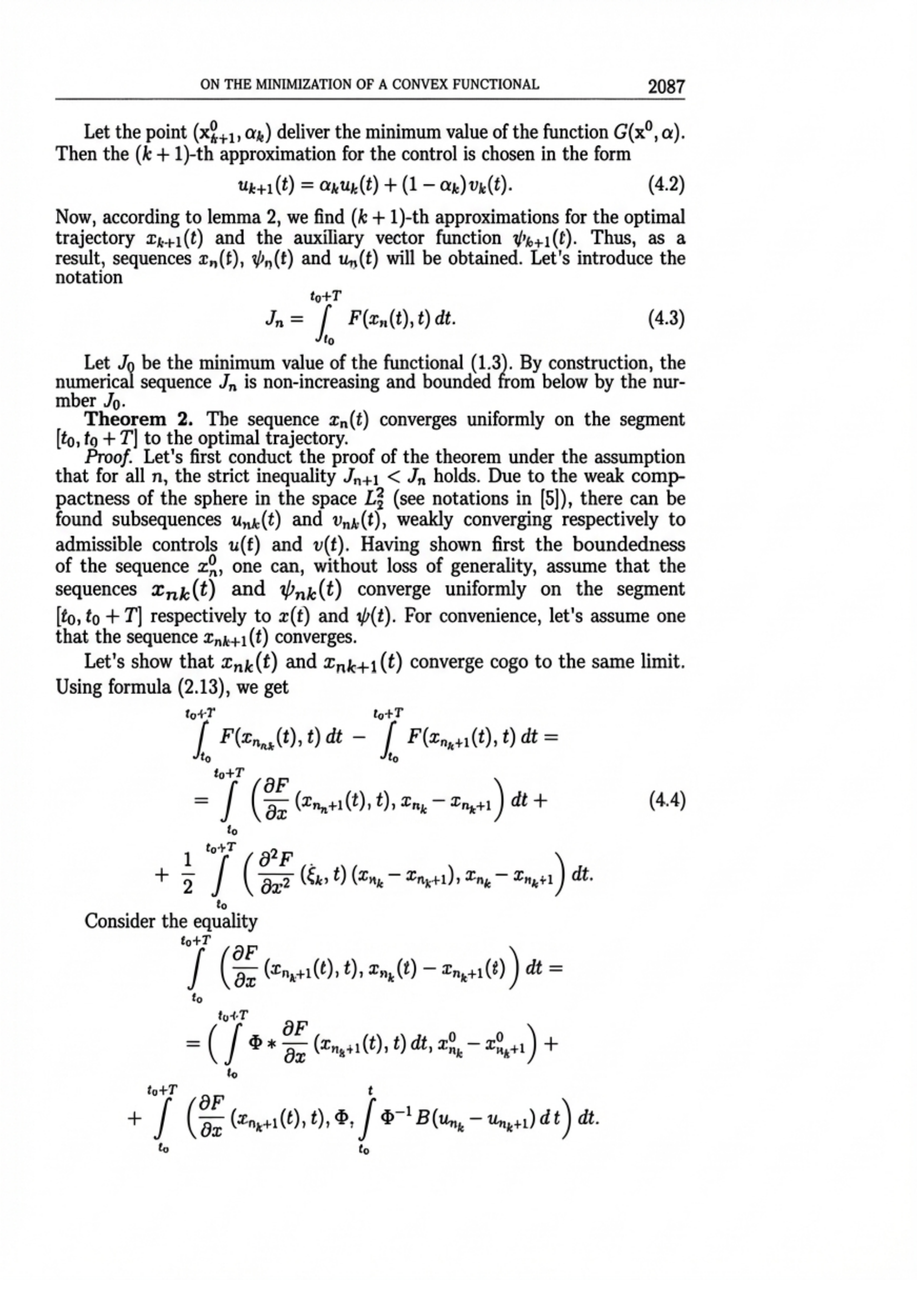
Let the k-th approximation $u_k(t)$, $x_k(t)$, $\psi_k(t)$ be constructed. Knowing $\psi_k(t)$, we find the function $v_k(t)$ from the condition

$$(\psi_k(t), B v_k(t)) = \max_{w_k U} (\psi_k(t), B u). \tag{4.1}$$

Applying lemma 3 with

$$w(\alpha, t) = w_k(\alpha, t) = \alpha\, u_k(t) + (1 - \alpha)\, v_k(t).$$

Figure 8: Figure 6



Let the point $(\mathbf{x}^0_{k+1}, \alpha_k)$ deliver the minimum value of the function $G(\mathbf{x}^0, \alpha)$. Then the $(k+1)$-th approximation for the control is chosen in the form

$$u_{k+1}(t) = \alpha_k u_k(t) + (1-\alpha_k) v_k(t). \tag{4.2}$$

Now, according to lemma 2, we find $(k+1)$-th approximations for the optimal trajectory $x_{k+1}(t)$ and the auxiliary vector function $\psi_{k+1}(t)$. Thus, as a result, sequences $x_n(t)$, $\psi_n(t)$ and $u_n(t)$ will be obtained. Let's introduce the notation

$$J_n = \int_{t_0}^{t_0+T} F(x_n(t), t)\, dt. \tag{4.3}$$

Let J_0 be the minimum value of the functional (1.3). By construction, the numerical sequence J_n is non-increasing and bounded from below by the number J_0.

Theorem 2. The sequence $x_n(t)$ converges uniformly on the segment $[t_0, t_0+T]$ to the optimal trajectory.

Proof. Let's first conduct the proof of the theorem under the assumption that for all n, the strict inequality $J_{n+1} < J_n$ holds. Due to the weak compactness of the sphere in the space L^2_2 (see notations in [5]), there can be found subsequences $u_{nk}(t)$ and $v_{nk}(t)$, weakly converging respectively to admissible controls $u(t)$ and $v(t)$. Having shown first the boundedness of the sequence x^0_n, one can, without loss of generality, assume that the sequences $x_{nk}(t)$ and $\psi_{nk}(t)$ converge uniformly on the segment $[t_0, t_0+T]$ respectively to $x(t)$ and $\psi(t)$. For convenience, let's assume one that the sequence $x_{nk+1}(t)$ converges.

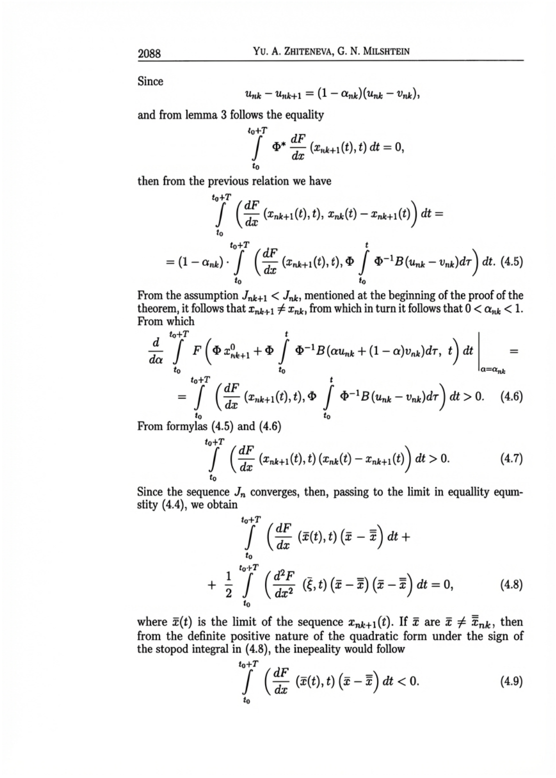
Let's show that $x_{nk}(t)$ and $x_{nk+1}(t)$ converge cogo to the same limit. Using formula (2.13), we get

$$\begin{aligned}\int_{t_0}^{t_0+T} F(x_{n_{nk}}(t), t)\, dt - \int_{t_0}^{t_0+T} F(x_{n_k+1}(t), t)\, dt &= \\ = \int_{t_0}^{t_0+T} \left(\frac{\partial F}{\partial x}(x_{n_n+1}(t), t), x_{n_k} - x_{n_k+1}\right) dt &+ \\ + \frac{1}{2}\int_{t_0}^{t_0+T} \left(\frac{\partial^2 F}{\partial x^2}(\dot{\xi}_k, t)(x_{n_k} - x_{n_k+1}), x_{n_k} - x_{n_k+1}\right) dt&.\end{aligned} \tag{4.4}$$

Consider the equality

$$\begin{aligned}\int_{t_0}^{t_0+T} \left(\frac{\partial F}{\partial x}(x_{n_k+1}(t), t), x_{n_k}(t) - x_{n_k+1}(t)\right) dt &= \\ = \left(\int_{t_0}^{t_0+T} \Phi * \frac{\partial F}{\partial x}(x_{n_k+1}(t), t)\, dt, x^0_{n_k} - x^0_{n_k+1}\right) &+ \\ + \int_{t_0}^{t_0+T} \left(\frac{\partial F}{\partial x}(x_{n_k+1}(t), t), \Phi, \int_{t_0}^{t} \Phi^{-1} B(u_{n_k} - u_{n_k+1})\, dt\right) dt&.\end{aligned}$$

Figure 9: Figure 7



Since

$$u_{nk} - u_{nk+1} = (1 - \alpha_{nk})(u_{nk} - v_{nk}),$$

and from lemma 3 follows the equality

$$\int\limits_{t_0}^{t_0+T} \Phi^* \frac{dF}{dx}\,(x_{nk+1}(t), t)\,dt = 0,$$

then from the previous relation we have

$$\int\limits_{t_0}^{t_0+T} \left(\frac{dF}{dx}\,(x_{nk+1}(t), t),\ x_{nk}(t) - x_{nk+1}(t)\right) dt =$$

$$= (1 - \alpha_{nk}) \cdot \int\limits_{t_0}^{t_0+T} \left(\frac{dF}{dx}\,(x_{nk+1}(t), t), \Phi \int\limits_{t_0}^{t} \Phi^{-1} B(u_{nk} - v_{nk})d\tau\right) dt. \quad (4.5)$$

From the assumption $J_{nk+1} < J_{nk}$, mentioned at the beginning of the proof of the theorem, it follows that $x_{nk+1} \neq x_{nk}$, from which in turn it follows that $0 < \alpha_{nk} < 1$. From which

$$\frac{d}{d\alpha} \int\limits_{t_0}^{t_0+T} F\left(\Phi\, x^0_{nk+1} + \Phi \int\limits_{t_0}^{t} \Phi^{-1} B(\alpha u_{nk} + (1-\alpha) v_{nk})d\tau,\ t\right) dt \bigg|_{\alpha = \alpha_{nk}} =$$

$$= \int\limits_{t_0}^{t_0+T} \left(\frac{dF}{dx}\,(x_{nk+1}(t), t), \Phi \int\limits_{t_0}^{t} \Phi^{-1} B(u_{nk} - v_{nk})d\tau\right) dt > 0. \quad (4.6)$$

From formylas (4.5) and (4.6)

$$\int\limits_{t_0}^{t_0+T} \left(\frac{dF}{dx}\,(x_{nk+1}(t), t)\,(x_{nk}(t) - x_{nk+1}(t)\right) dt > 0. \quad (4.7)$$

Since the sequence J_n converges, then, passing to the limit in equallity equmstity (4.4), we obtain

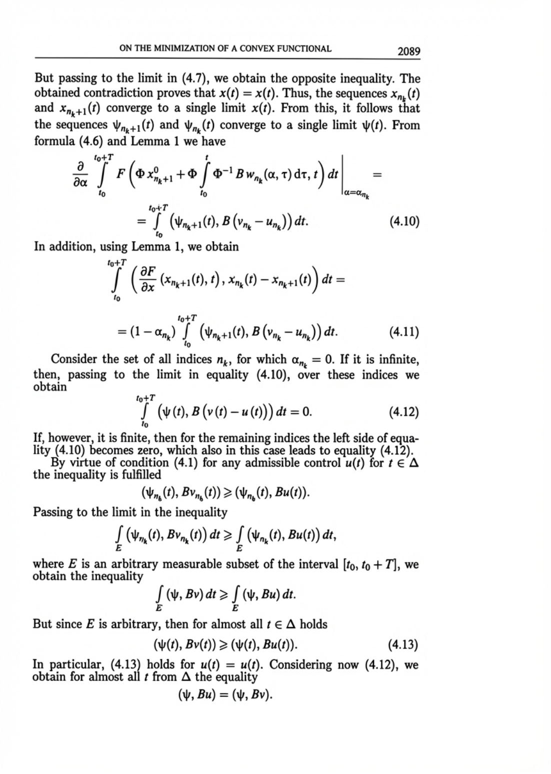
$$\int\limits_{t_0}^{t_0+T} \left(\frac{dF}{dx}\,(\bar{x}(t), t)\,(\bar{x} - \bar{\bar{x}}\right) dt +$$

$$+ \frac{1}{2} \int\limits_{t_0}^{t_0+T} \left(\frac{d^2F}{dx^2}\,(\bar{\xi}, t)\,(\bar{x} - \bar{\bar{x}})\,(\bar{x} - \bar{\bar{x}}\right) dt = 0, \quad (4.8)$$

where $\bar{x}(t)$ is the limit of the sequence $x_{nk+1}(t)$. If $\bar{x}$ are $\bar{x} \neq \bar{\bar{x}}_{nk}$, then from the definite positive nature of the quadratic form under the sign of the stopod integral in (4.8), the inepeality would follow

$$\int\limits_{t_0}^{t_0+T} \left(\frac{dF}{dx}\,(\bar{x}(t), t)\,(\bar{x} - \bar{\bar{x}}\right) dt < 0. \quad (4.9)$$

Figure 10: Figure 8



But passing to the limit in (4.7), we obtain the opposite inequality. The obtained contradiction proves that $x(t) = x(t)$. Thus, the sequences $x_{n_k}(t)$ and $x_{n_k+1}(t)$ converge to a single limit $x(t)$. From this, it follows that the sequences $\psi_{n_k+1}(t)$ and $\psi_{n_k}(t)$ converge to a single limit $\psi(t)$. From formula (4.6) and Lemma 1 we have

$$\frac{\partial}{\partial\alpha}\int_{t_0}^{t_0+T} F\left(\Phi x^0_{n_k+1} + \Phi\int_{t_0}^{t}\Phi^{-1} B w_{n_k}(\alpha,\tau)\,d\tau, t\right) dt\,\Bigg|_{\alpha=\alpha_{n_k}} =$$

$$= \int_{t_0}^{t_0+T}\left(\psi_{n_k+1}(t), B\left(v_{n_k} - u_{n_k}\right)\right) dt. \tag{4.10}$$

In addition, using Lemma 1, we obtain

$$\int_{t_0}^{t_0+T}\left(\frac{\partial F}{\partial x}\left(x_{n_k+1}(t), t\right), x_{n_k}(t) - x_{n_k+1}(t)\right) dt =$$

$$= (1-\alpha_{n_k})\int_{t_0}^{t_0+T}\left(\psi_{n_k+1}(t), B\left(v_{n_k} - u_{n_k}\right)\right) dt. \tag{4.11}$$

Consider the set of all indices n_k, for which $\alpha_{n_k} = 0$. If it is infinite, then, passing to the limit in equality (4.10), over these indices we obtain

$$\int_{t_0}^{t_0+T}\left(\psi(t), B\left(v(t) - u(t)\right)\right) dt = 0. \tag{4.12}$$

If, however, it is finite, then for the remaining indices the left side of equality (4.10) becomes zero, which also in this case leads to equality (4.12).

By virtue of condition (4.1) for any admissible control $u(t)$ for $t \in \Delta$ the inequality is fulfilled

$$(\psi_{n_k}(t), Bv_{n_k}(t)) \geqslant (\psi_{n_k}(t), Bu(t)).$$

Passing to the limit in the inequality

$$\int_E \left(\psi_{n_k}(t), Bv_{n_k}(t)\right) dt \geqslant \int_E \left(\psi_{n_k}(t), Bu(t)\right) dt,$$

where E is an arbitrary measurable subset of the interval $[t_0, t_0 + T]$, we obtain the inequality

$$\int_E (\psi, Bv)\, dt \geqslant \int_E (\psi, Bu)\, dt.$$

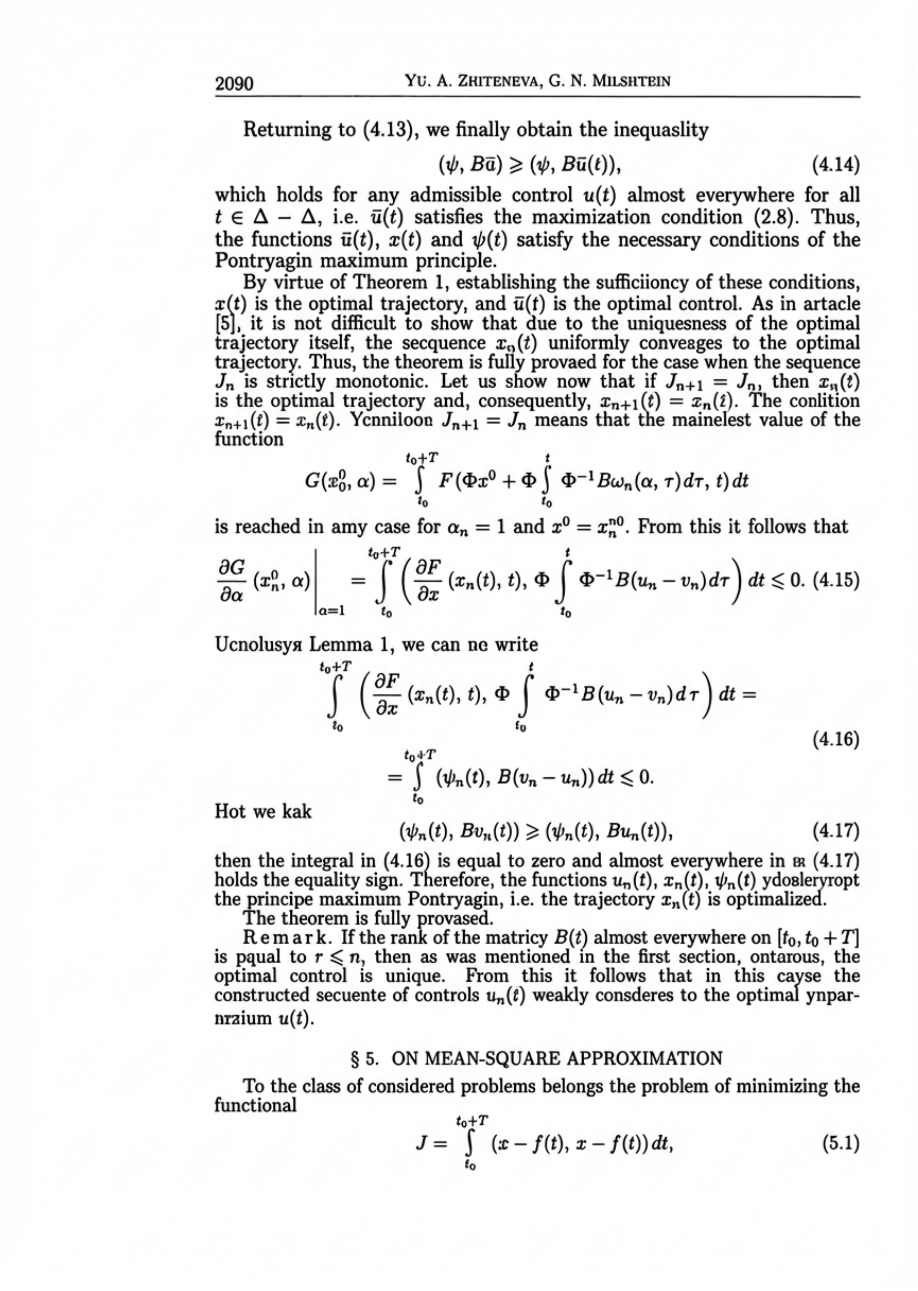
But since E is arbitrary, then for almost all $t \in \Delta$ holds

$$(\psi(t), Bv(t)) \geqslant (\psi(t), Bu(t)). \tag{4.13}$$

In particular, (4.13) holds for $u(t) = u(t)$. Considering now (4.12), we obtain for almost all t from Δ the equality

$$(\psi, Bu) = (\psi, Bv).$$

Figure 11: Figure 9



Returning to (4.13), we finally obtain the inequaslity

$$(\psi,\, B\bar{u}) \geqslant (\psi,\, B\bar{u}(t)), \tag{4.14}$$

which holds for any admissible control $u(t)$ almost everywhere for all $t \in \Delta - \Delta$, i.e. $\bar{u}(t)$ satisfies the maximization condition (2.8). Thus, the functions $\bar{u}(t)$, $x(t)$ and $\psi(t)$ satisfy the necessary conditions of the Pontryagin maximum principle.

By virtue of Theorem 1, establishing the sufficiioncy of these conditions, $x(t)$ is the optimal trajectory, and $\bar{u}(t)$ is the optimal control. As in artacle [5], it is not difficult to show that due to the uniquesness of the optimal trajectory itself, the secquence $x_n(t)$ uniformly conveages to the optimal trajectory. Thus, the theorem is fully provaed for the case when the sequence J_n is strictly monotonic. Let us show now that if $J_{n+1} = J_n$, then $x_n(t)$ is the optimal trajectory and, consequently, $x_{n+1}(t) = x_n(t)$. The conlition $x_{n+1}(t) = x_n(t)$. Ycnniloon $J_{n+1} = J_n$ means that the mainelest value of the function

$$G(x_0^0,\, \alpha) = \int\limits_{t_0}^{t_0+T} F\Big(\Phi x^0 + \Phi \int\limits_{t_0}^{t} \Phi^{-1} B\omega_n(\alpha,\, \tau)\, d\tau,\, t\Big)\, dt$$

is reached in amy case for $\alpha_n = 1$ and $x^0 = x_n^{n0}$. From this it follows that

$$\frac{\partial G}{\partial \alpha}\,(x_n^0,\, \alpha)\bigg|_{\alpha=1} = \int\limits_{t_0}^{t_0+T} \left(\frac{\partial F}{\partial x}\,(x_n(t),\, t),\, \Phi \int\limits_{t_0}^{t} \Phi^{-1} B(u_n - v_n)\, d\tau\right) dt \leqslant 0. \tag{4.15}$$

Ucnolusyя Lemma 1, we can ne write

$$\begin{gathered}\int\limits_{t_0}^{t_0+T} \left(\frac{\partial F}{\partial x}\,(x_n(t),\, t),\, \Phi \int\limits_{t_0}^{t} \Phi^{-1} B(u_n - v_n)\, d\tau\right) dt = \\ = \int\limits_{t_0}^{t_0+T} (\psi_n(t),\, B(v_n - u_n))\, dt \leqslant 0.\end{gathered} \tag{4.16}$$

Hot we kak

$$(\psi_n(t),\, Bv_n(t)) \geqslant (\psi_n(t),\, Bu_n(t)), \tag{4.17}$$

then the integral in (4.16) is equal to zero and almost everywhere in вr (4.17) holds the equality sign. Therefore, the functions $u_n(t)$, $x_n(t)$, $\psi_n(t)$ ydoвleryropt the principe maximum Pontryagin, i.e. the trajectory $x_n(t)$ is optimalized.

The theorem is fully provased.

Remark. If the rank of the matricy $B(t)$ almost everywhere on $[t_0, t_0 + T]$ is pqual to $r \leqslant n$, then as was mentioned in the first section, ontarous, the optimal control is unique. From this it follows that in this cayse the constructed secuente of controls $u_n(t)$ weakly consderes to the optimal ynparnrzium $u(t)$.

§ 5. ON MEAN-SQUARE APPROXIMATION

To the class of considered problems belongs the problem of minimizing the functional

$$J = \int\limits_{t_0}^{t_0+T} (x - f(t),\, x - f(t))\, dt, \tag{5.1}$$

Figure 12: Figure 10