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P. P. MOSOLOV, V. P. MYASNIKOV

ON RECTILINEAR STATIONARY MOTIONS OF A VISCO-PLASTIC MEDIUM

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As is known, the mathematical difficulties associated with the consideration of problems on the motion of a visco-plastic medium consist in the fact that boundary-value problems for nonlinear partial differential equations corresponding to this medium must be studied in domains with unknown boundaries, which are the boundaries of plastic flow. In this case not only the question of the qualitative features of the solutions, but also the question of their existence and uniqueness, is very complicated. In connection with the indicated difficulties, in this work a variational method is used to study the properties of a model of a visco-plastic medium, which, as it seems to us, possesses a number of advantages in comparison with the local (differential) method, based on differential equations.

The variational principle, formulated in papers ([1–3]), in the case of rectilinear stationary motions of a visco-plastic medium is as follows. The actual stationary motion of an incompressible visco-plastic medium differs from any kinematically possible motion, with an unchanged flow of energy through the boundary Γ of the volume ω occupied by the medium, in that it minimizes the functional

$$I(u) = \int\limits_{\omega} \left\{ \frac{\mu}{2} \, |\nabla u|^2 + \tau_0 |\nabla u| \right\} d\omega - \int\limits_{\omega} X u \, d\omega - \int\limits_{\Gamma} Q u \, d\Gamma, \qquad (1)$$

where μ is the coefficient of viscosity of the medium; τ_0 is the yield limit; X, Q are external body and surface forces; $u(x, y)$ is the velocity of the medium; x, y are a Cartesian coordinate system in the plane perpendicular to the direction of motion of the particles.

The first question that arises—the correctness of the formulation of the problem—is resolved here with the aid of a simple theorem of functional analysis: if a strictly convex, increasing, continuous functional is defined on a closed convex set of a reflexive Banach space, then this functional attains its least value on the set under consideration.

Let us consider the problem of the stationary motion of a visco-plastic medium in a cylindrical pipe under the action of a constant pressure gradient C. Denote by ω the cross-section of the pipe. The stationary motion is characterized by the distribution of velocities $u(x, y)$ in the direction of the pipe axis in the domain ω. Functional (1) in this case has the form

$$I(u) = \int\limits_{\omega} \left\{ \frac{\mu}{2} |\nabla u|^2 + \tau_0 |\nabla u| - Cu \right\} d\omega. \tag{2}$$

On the boundary of the domain ω we impose the no-slip condition $u|_{\Gamma} = 0$. The true stationary motion is distinguished among all kinematically possible motions by the fact that functional (2) attains its minimum on it.

The motion of a visco-plastic medium in a pipe possesses the following qualitative features: 1) the onset of motion; 2) flow cores; 3) stagnant flow zones.

1) A characteristic property of a viscoplastic medium is the impossibility of the existence of a flow for small pressure gradients. Namely, the following assertion holds: there exists a critical value of the pressure gradient C^* such that, for $C < C^*$, the flow is absent, i.e. $u = 0$; for $C > C^*$, the flow exists. The quantity C^* is determined by the equality

$$C^* = \tau_0 \Big/ \sup_{\omega' \subseteq \omega} \frac{\operatorname{mes} \omega'}{\operatorname{mes} \Gamma'}.$$

Here ω' is an arbitrary subdomain of the domain ω; Γ' is the boundary of ω'. It can be proved that there exists a subdomain ω_1 in ω, with boundary Γ_1, such that

$$\sup_{\omega' \subseteq \omega} \frac{\operatorname{mes} \omega'}{\operatorname{mes} \Gamma'} = \frac{\operatorname{mes} \omega_1}{\operatorname{mes} \Gamma_1},$$

where the boundary Γ_1 is constructed as follows: the parts of Γ_1 lying inside ω are arcs of circles tangent to the boundary Γ. In what follows, the curve Γ_1 will be called the **contraction contour**. The indicated structure of contraction contours makes it possible to compute effectively the quantities C^* in the case when ω is a polygon. For example, if ω is an equilateral triangle with side a, then

$$C^* = \frac{2\tau_0}{a} \left(\sqrt{3} + \sqrt{\pi/3} \right).$$

2) We shall assume that $C > C^*$. The following assertion holds. The minimizing function $u(x, y)$ is a positive, continuous function having no local minima inside ω. If the domain ω is p-connected, then $u(x, y)$ attains each of its local maxima on a domain containing a disk of radius $\tau_0/8pC$ and not containing a disk of radius greater than $2\tau_0/C$. Each local maximum of the function $u(x, y)$ is called a **flow core**. Obviously, if a stationary flow of a viscoplastic medium exists, then it has at least one flow core. The lower estimate just given for the flow core can be substantially improved if ω is simply connected. Namely, in this case, every region of a local maximum contains a disk of radius greater than τ_0/C. The upper and lower estimates for the flow core in a simply connected domain are sharp. The upper estimate is attained for a circular tube, and the lower one for a plane-parallel gap.

Let us also indicate the following important characteristics of the flow core. Let A be the subdomain of the domain ω corresponding to the flow core; let a be the boundary of A. Then $\tau_0 \operatorname{mes} a = C \operatorname{mes} A$ ($\operatorname{mes} a$ is the length of the boundary a, and $\operatorname{mes} A$ is the area of the domain A). Let A' be an arbitrary subdomain of the domain A, and let a' be the boundary of A'. Then

$$\sup \frac{\operatorname{mes} A'}{\operatorname{mes} a'} = \frac{\operatorname{mes} A}{\operatorname{mes} a}.$$

These properties make it possible to refine the geometric structure of the cores. For example, it follows from them that a core cannot contain angular points directed toward the flow. Physically, the latter assertions are equivalent to the condition that the core moves as a rigid body.

Let us indicate a sufficient condition for the flow core to be unique. Consider a simply connected domain ω that is mapped onto itself by rotation through an angle π/n, $n \geq 2$. This domain has n axes of symmetry. Consider a line L parallel to one of the axes of symmetry ol. Suppose that the line L divides the domain ω into subdomains ω' and ω'' in such a way that the center of symmetry of ω lies in ω'. The domain ω is called expanding in the direction ol if, for any line L with the indicated properties, the domain ω^*, symmetric to ω'' with respect to the line L, lies in the domain ω'. If the domain ω, expanding in the directions of the axes $ol_1, \dots, ol_n$, $n \geq 2$, then the flow has only one core. We note that among the expanding domains under consideration there are nonconvex domains.

3. A **stagnation zone** is a subdomain of the domain ω, adjacent to the boundary Γ, in which the velocity of motion is equal to zero. Let

$$\tau \Big/ \left(\frac{\operatorname{mes} \omega}{\operatorname{mes} \Gamma}\right) > C > C^* > \tau_0 \Bigg/ \left(\sup_{\omega' \subset \omega} \frac{\operatorname{mes} \omega'}{\operatorname{mes} \Gamma'}\right). \tag{3}$$

Then in the domain ω a flow exists, and a stagnation zone exists. We note that condition (3) makes it possible to detect the existence of a stagnation zone when the pressure gradient C differs little from C^*. It is not difficult to see that for large pressure gradients a stagnation zone may not exist at all. Thus, for example, if every point of the boundary of the domain ω can be touched from within by a circle of radius greater than $2\tau_0/C$, then there are no stagnation zones in the domain ω. However, for example, in a square a stagnation zone exists for any values of the pressure gradient. The form and magnitude of the stagnation zone are known in the case when the domain is an obtuse angle (4). Place the domain ω inside this obtuse angle. Then the intersection of the stagnation zone with the domain ω gives a part of the stagnation zone in ω. This method makes it possible to estimate from below the magnitude of the stagnation zone. In an analogous way one can indicate conditions under which the stagnation zone has the character of a bridge between several flow regions (dumbbell-shaped regions). For stagnation zones one can also give an estimate of their magnitude from above. Suppose that in the domain ω there exists only one yielding contour Γ_1. Then the stagnation zone is always enclosed between the boundary of the domain Γ and the yielding contour Γ_1. Let us note one more important geometric property of stagnation zones. The part of the boundary of a stagnation zone that does not coincide with the boundary Γ is concave with respect to the region of the plastic state of the medium, i.e., the segment joining two sufficiently close points of this part of the boundary lies in the region of plastic flow.

Consider the problem of the stationary motion of a system of cylinders with mutually parallel generators, immersed in a viscoplastic medium under the action of forces applied to these cylinders and acting in the direction of the axes of the cylinders. Let a cylindrical tube with cross-section ω be filled with a viscoplastic medium, and let ω_i be the cross-sections of the cylinders immersed in this medium ($i = 1, \dots, N$). Denote by R_i the force applied to the i-th cylinder. The functional (1) is rewritten in the form

$$\int_\Omega \left\{ \frac{\mu}{2} |\nabla u|^2 + \tau |\nabla u| \right\} d\omega - \sum_1^N R_i u|_{\Gamma_i}, \qquad u|_\Gamma = 0, \tag{4}$$

where $\Omega = \omega \setminus \bigcup_1^N \omega_i$, Γ_i is the boundary of ω_i, $u|_{\Gamma_i} = a_i$ is constant (the boundary conditions correspond to no-slip conditions), and $u(x,y)$ is the velocity of the particles of the medium in the direction of the tube axis.

In this problem there occur effects analogous to the effects obtained in the preceding problem. Namely: 1) the onset of motion; 2) a finite region of propagation of disturbances in the medium during the motion of the cylinders; 3) entrainment of the medium by the cylinders.

1) In order for motion to be absent, it is necessary and sufficient that

$$\tau_0 \operatorname{mes} L_{i_1 \dots i_k} \geq \sum_{p=1}^{k} R_{i_p},$$

where $L_{i_1 \dots i_k}$ is a closed curve in Ω enclosing only the regions $\omega_{i_1}, \dots, \omega_{i_k}$ and having, in the class of all closed curves in Ω homotopic to it, the least length. By $\operatorname{mes} L_{i_1 \dots i_k}$ is denoted the length of the curve $L_{i_1 \dots i_k}$. We note that for any distribution of the quantities R_i on ω_i and any distribution of the regions ω_i inside ω, motion exists if

$$\sum_{1}^{N} R_i > \tau_0 \operatorname{mes} \Gamma.$$

2) For simplicity, consider the case when in ω there is only one cavity ω_1 ($N = 1$). In this case the condition for the existence of the motion is the following: $\tau_0 \operatorname{mes} L_1 < R_1$. It is not difficult to verify that the domain ω_1 can be placed inside a disk K of radius $R_1/4\tau_0$. Let the domain ω contain the disk K. Then the zone of disturbance of the medium lies entirely inside this disk, i.e., outside K, $u(x, y) \equiv 0$.

3) If $\tau_0 \operatorname{mes} L_1 < R_1 < \tau_0 \operatorname{mes} \Gamma_1$, then the motion exists and part of the medium moves as a single whole together with the cylinder. We note that in the case of one cylinder the effect of entrainment of the medium can occur only if ω_1 is a nonconvex domain.

Consider an example. Let there be two cylinders in the domain ω, with identical square cross sections ω_1, ω_2. Suppose that ω_1, ω_2 are positioned so that by a translation parallel to one of the sides the square ω_1 can be carried into the square ω_2. Let a be the side of the square, and d the distance between the squares. Suppose that $a > d$ and $R_1 < 4\tau_0 a,\ R_2 < 4\tau_0 a,\ R_1 + R_2 > (6a + 2d)\tau_0$. Then the motion in ω exists and the cylinders move as one whole, entraining a part of the medium located between them.

We note that all the results set forth above were obtained from consideration of the properties of functions minimizing the functionals (2), (4), without using the Euler equation for these functionals.

The model of a viscoplastic medium contains two limiting cases, $\mu = 0$ and $\tau_0 = 0$. The latter case is well known (a viscous fluid, slow motion). Consider the limiting transition as $\mu \to 0$. If one requires that as $\mu \to 0$ the velocities of motion remain finite, then it necessarily follows from this that the external loads must tend to the critical ones, while the values of the functionals (2) and (4) tend to zero. Since in the limiting transition as $\mu \to 0$ two variable quantities take part, μ and the external load, in the limit at $\mu = 0$ we obtain different functions giving stationary motions of an ideally plastic medium.

Thus, without additional assumptions concerning the relation between the change in the external load and the viscosity, it is impossible to single out a unique stationary motion of an ideally plastic medium.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.