ON LINEAR DIFFERENTIAL GAMES. 2

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Abstract

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Here we shall develop the considerations set forth in (^{2}), both in the direction of extending the class of games under consideration and in the direction of improving the result.

Let R be a vector space of arbitrary dimension, and

$$dz/dt = Cz + u - v \tag{1}$$

a vector differential equation given in R, so that $z \in R$; here C is a square matrix, and $u \in P,\ v \in Q$ are control parameters, with P and Q closed bounded convex subsets of the space R of arbitrary dimension. Further, let M be a closed convex subset of the space R, also of arbitrary dimension. Equation (1) and the set M define a differential game. Here u is the pursuing parameter, v the evading parameter, and M the set on which the game ends (^{1}).

For the formulation and proof of the result we introduce some operations on convex sets, in particular the operation of alternating integration of convex sets. All the sets considered below are closed subsets of the space R.

A. Let A and B be two sets, and let α and β be two real numbers. By

$$\alpha A + \beta B \tag{2}$$

we denote the totality of all vectors $\alpha x + \beta y$, where $x \in A,\ y \in B$. In the case $\alpha = \beta = 1$, formula (2) gives the algebraic sum of sets. In the case $\alpha = 1,\ \beta = -1$, formula (2) gives the algebraic difference of sets. If A and B are convex sets, formula (2) gives convex sets.

By

$$A \mathbin{\underline{*}} B \tag{3}$$

we denote the totality of all vectors x satisfying the condition $x + B \subset A$. The set (3) may also turn out to be empty. If A is a convex set, then formula (3) also defines a convex set. Let A, U, V be convex sets. Then the following easily proved relations hold:

$$(A \underline{*} U) \underline{*} V = A \underline{*} (U + V), \tag{4}$$

$$(A + U) \underline{*} V \supset (A \underline{*} V) + U. \tag{5}$$

B. Let A_0 be some convex set, and

$$U_1, \dots, U_n; \qquad V_1, \dots, V_n \tag{6}$$

two sequences of convex sets. Define inductively the set A_{i+1}, $i = 0, 1, \dots, n-1$, by putting

$$A_{i+1} = (A_i + U_{i+1}) \underline{*} V_{i+1}. \tag{7}$$

The set A_n is naturally called the alternating sum of the sequences (6) with initial value A_0. Let

$$U = U_1 + \dots + U_n; \qquad V = V_1 + \dots + V_n,$$

then from formulas (4) and (5) it follows that

$$A_n \subset (A_0 + U) \underline{*} V. \tag{8}$$

B. Let $A = A_0$ be some convex set, and let $U(\tau)$ and $V(\tau)$ be two bounded convex sets depending continuously on the real parameter τ on the interval $p \leq \tau \leq q$. Define the alternating integral of the functions $U(\tau)$ and $V(\tau)$:

$$B = \int_{A,p}^{q} [U(\tau) d\tau \underline{*} V(\tau) d\tau]. \tag{9}$$

Here A is the initial set of integration, p is the initial value of τ, and q is the final value. The integral (9) itself is a convex set. To define the integral (9), divide the interval of integration into small subintervals by the points $r_0 = p, r_1, \dots, r_n = q$, and let

$$U_i = \int_{r_{i-1}}^{r_i} U(\tau) d\tau; \qquad V_i = \int_{r_{i-1}}^{r_i} V(\tau) d\tau \quad (i = 1, \dots, n). \tag{10}$$

The integrals of convex sets written here are defined in the natural way on the basis of the addition operation (2).

Starting from the sequences (10) and the initial set A_0, we construct the alternating sum A_n (see B). The limit of this alternating sum under unlimited refinement of the interval $p \leq \tau \leq q$ is the integral (9). Suppose now that the functions $U(\tau)$ and $V(\tau)$ are defined on the interval $p \leq \tau \leq r$, with $r > q$; then the inclusion holds

$$\int_{A,p}^{r} [U(\tau)d\tau \underset{-}{*} V(\tau)d\tau] \subset \left(B + \int_{q}^{r} U(\tau)d\tau\right) \underset{-}{*} \int_{q}^{r} V(\tau)d\tau \tag{11}$$

(see (8), (9)).

Theorem. Put $A = -M$ (see (1)) and form the alternating integral

$$W(\tau) = \int_{A,0}^{\tau} [e^{rC} P\, dr \underset{-}{*} e^{rC} Q\, dr] \quad (\tau \geq 0). \tag{12}$$

Further, let z_0 be an arbitrary point of the space R not belonging to M; put $\eta(\tau) = e^{\tau C} z_0$. If, for some value $\tau > 0$, the point $-\eta(\tau)$ belongs to the convex set $W(\tau)$, then denote by τ_0 the minimal value of τ for which this membership is realized. It then turns out that the game (1), beginning at the point z_0, can be ended in a time not exceeding the number

$$T(z_0) = \tau_0. \tag{13}$$

In proving this theorem, the control u will be constructed with account taken of the control v, so as to shorten the time of the game as much as possible. In constructing the control $u(t)$ at the time t, we shall use the value $z(t)$ at the same time and the control $v(s)$ on the interval $t \leq s \leq t + \varepsilon$, where ε is an arbitrarily small positive number (see (?)).

Proof. We shall assume that the control $v(t)$ is prescribed on the interval $0 \leq t \leq \varepsilon$, and let $u(t)$ for the moment be an arbitrary control prescribed on the same interval. Substituting these controls into equation (1), we find its solution $z(t)$ on the interval $0 \leq t \leq \varepsilon$ under the initial condition $z(0) = z_0$. The number $T(z(\varepsilon))$ (see (13)) is a functional of the function

$u(t)$. Below we shall choose the function $u(t)$ in such a way that the number $T(z(\varepsilon))$ assumes its minimal value, and we shall prove that

$$T(z_0) - T(z(\varepsilon)) \geqslant \varepsilon. \tag{14}$$

It follows from (11) that, for arbitrary $\tau \geqslant \varepsilon$, the following inclusion holds:

$$W(\tau) \subset \left(W(\tau - \varepsilon) + \int_{\tau-\varepsilon}^{\tau} e^{rC} P\, dr\right) * \int_{\tau-\varepsilon}^{\tau} e^{rC} Q\, dr \subset$$

$$\subset W(\tau-\varepsilon)+\int_{\tau-\varepsilon}^{\tau} e^{rC}P\,dr-\int_{\tau-\varepsilon}^{\tau} e^{rC}v(\tau-r)\,dr=D(\tau). \tag{15}$$

It should be noted that the last term of formula (15) is defined for all values $\tau > \varepsilon$, since the function $v(\tau-r)$ is defined on the whole interval of integration $\tau-\varepsilon \leqslant r \leqslant \tau$, for its argument on this interval of integration varies over the interval $[0,\varepsilon]$, and the function $v(t)$ is specified on this interval. By assumption, the point $-\eta(\tau)$ belongs to the left-hand side of inclusion (15) for $\tau=\tau_0$. Let $\tau_1 \leqslant \tau_0$ be the minimal value of τ for which the point $-\eta(\tau)$ belongs to the last part of the inclusion, namely to the set $D(\tau)$. Then there exists a function $u(t) \in P$, $0 \leqslant t \leqslant \varepsilon$, such that the point $-e^{(\tau_1-\varepsilon)C}z_1$ belongs to the set $W(\tau_1-\varepsilon)$, under the condition that

$$z_1=e^{\varepsilon C}z_0+\int_0^{\varepsilon} e^{sC}\big(u(\varepsilon-s)-v(\varepsilon-s)\big)\,ds. \tag{16}$$

Since, evidently, $z_1=z(\varepsilon)$, the assertion is proved.

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