On the smoothness of thermal potentials. IV

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Abstract

The paper considers the application of Pagnis'theory of the special heat potential of a simple layer to a boundary value problem for a system of parabolic equations with discontinuous coefficients, arising in the study of the concentration distribution of substances involved in the vital processes of a living cell. The boundary conditions and matching conditions on the discontinuity surfaces of this problem include derivatives of the solution along oblique directions. The study of the smoothness of Pagnis' special heat potential of a simple layer, depending on the smoothness of its density distributed on non-cylindrical surfaces of type $^{1,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$ and $^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}$, allows for proving the existence of a solution from the class $H^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}$ (in the domains of continuity of the solution) for the problem under minimum admissible smoothness requirements for the problem data. The proof of the existence theorem is carried out using the classical method of continuation with respect to a parameter, utilizing a $(2+\alpha)$ Schauder-type a priori estimate established for the solution of the boundary value problem under consideration. (The derivation of this estimate is performed using the method of the works of the author and V. N. Maslennikova, who obtained a $(2+\alpha)$ a priori estimate for the solution of the II and III boundary value problems with an oblique derivative in non-cylindrical domains for a second-order parabolic equation). Vyborny' s theorem on the sign of the oblique derivative of a solution to a parabolic equation at a boundary extremum point, as well as the research of the author and V. N. Maslennikova on the applications of the maximum principle for parabolic equations with discontinuous coefficients, allow for specifying the conditions under which the solution to the boundary value problem is unique and admits an a priori estimate of the modulus. Bibliography: 13.

Full Text

Preamble

This section continues the analysis of the boundary value problems established in Section 18 of the previous work (1967, Vol. III, No. 8, Part IV). Specifically, we consider the systems of equations and boundary conditions defined by (18.30), (18.4), (18.5X), and (18.6!). We denote the corresponding operators and functions as $a(D_T^0)$ for (18.30), (18.4), (18.5r), and (18.6r). Here, the indices $l = 1, 2$ and $k = 1, 2, \dots, m$ correspond to the components of the vector functions $f_{kl}(x,t)$ and $f_{kl}(x)$ defined in the respective domains.

For the boundary $\Gamma^{(2)}$, we assume the condition $\det \|a_{ij}^{(k)}(x,t)\| \neq 0$ for $(x,t) \in \Gamma^{(2)}$ according to (18.10). Similarly, for $\Gamma^{(1)}$, the condition $\det \|a_{ij}\| \neq 0$ holds for $(x,t) \in \Gamma^{(1)}$ as per (18.8). These conditions ensure the non-degeneracy of the systems under consideration. The functions $u_l(x,t)$ and their derivatives are analyzed within the Hölder spaces $H^{2+\alpha,1+\alpha/2}$ over the domains $D_T^{(s)}$ $(s = 1, 2)$.

Potential Representations and Integral Equations

Following the methodology in §10 of [4], we introduce the potentials for the solutions. Let $v(y,\tau)$ be a density function satisfying the conditions in (18.15). We define the surface potentials as follows:

$$[\phi_{sl}] = P(x,t) = \int_{\Gamma^{(s)}} \int_0^t P^{(sl)}(x,t;y,\tau)\phi_{sl}(y,\tau)\, d\sigma_y\, d\tau, \tag{21.2}$$

$$Q^{(s)}[\phi_{sl}] = Q^{(s)}(x,t) = \int_{\Gamma^{(s)}} \int_0^t Q^{(sl)}(x,t;y,\tau)\phi_{sl}(y,\tau)\, d\sigma_y\, d\tau. \tag{21.3}$$

In these expressions, the kernels $P^{(sl)}$ and $Q^{(sl)}$ are constructed using the fundamental solutions of the parabolic operators. The densities ϕ_{sl} are determined by the boundary conditions on $\Gamma^{(s)}$.

By applying the jump relations for these potentials (analogous to those in §11 of [4]), we obtain a system of integral equations for the unknown densities. Specifically, for $(x,t) \in \Gamma^{(s)}$, the limiting values of the operators satisfy:

$$Q^{(sl)}(x,t) = \bar{Q}^{(sl)}(x,t) \pm \frac{1}{2}(\det \|a_{ij}\|)^{-1/2}\phi_{sl}(x,t), \tag{21.4}$$

where $\bar{Q}^{(sl)}$ denotes the direct value of the potential on the surface. Substituting these into the boundary conditions (18.110)-(18.14) leads to the following system for ϕ_{sl}:

$$\begin{aligned} u_1(x,t) &= P^{(11)}[\phi_{11}] + P^{(12)}[\phi_{12}] + P^{(22)}[\phi_{22}], \\ u_2(x,t) &= P^{(12)}[\phi_{12}] + P^{(22)}[\phi_{22}]. \end{aligned} \tag{21.5}$$

The explicit forms of the equations for the densities ϕ_{22} and ϕ_{12} on the respective boundaries $\Gamma^{(2)}$ and $\Gamma^{(1)}$ are given by:

$$\phi_{22}(x,t) = 2^{1-n}\pi^{-n/2} \dots [-f^{(2)}(x,t) + \dots], \tag{21.6}$$

$$\phi_{12}(x,t) = 2^{1-n}\pi^{-n/2} \dots [-f^{(2)}(x,t) + \dots]. \tag{21.8}$$

These equations are of Volterra type with weakly singular kernels. Under the assumed smoothness of the coefficients and boundary data, these systems possess unique solutions in the appropriate Hölder spaces.

§ 22. Schauder-type Estimates and Existence Theorems

We now establish the primary a priori estimates for the solutions $u(x,t)$ in the Hölder spaces $H^{2+\alpha,1+\alpha/2}(D_T)$. We assume that the coefficients of the operators and the boundary data satisfy the following regularity conditions:

$$\max |a_{ij}^{(k)}|_{H^\alpha} + |b_i^{(k)}|_{H^\alpha} + |c^{(k)}|_{H^\alpha} \leq M_1, \tag{22.2}$$

$$|f_{kl}|_{H^\alpha} \leq M_3. \tag{22.3}$$

Under these assumptions, the solution $u_{kl}(x,t)$ satisfies the estimate:

$$|u_{kl}|_{H^{2+\alpha}} \leq C \left(\sum |f_{kl}|_{H^\alpha} + ... \right), \tag{22.4}$$

where the constant C depends on the ellipticity constants, the domains, and the M_i bounds, but is independent of the specific solution.

The proof of these estimates relies on localizing the problem near the boundaries $\Gamma^{(l)}$ and using the properties of the potentials defined in (21.2) and (21.3). For interior points, we utilize standard interior estimates for parabolic equations. Near the boundary, we transform the domain to a half-space and apply the estimates for the model problem:

$$L^{(0)}(v_i) = f_i(x,t), \quad (x,t) \in D_T^{(0)}, \tag{22.7}$$

$$v_i(x,0) = f_i^{(1)}(x), \quad x \in \Omega^{(0)}. \tag{22.8}$$

The boundary conditions on the hyperplane $x_n = 0$ take the form:

$$(-1)^s \frac{\partial v^{(0s)}}{\partial x_n} + \cdots = f_s^{(3)}(x,t). \tag{22.9}$$

Applying the results from [10] and [12], we obtain the required Hölder continuity for the densities ϕ_{sl} and subsequently for the solutions u_{kl}. Specifically, the estimates (22.13) and (22.14) confirm that the solutions belong to $H^{2+\alpha,1+\alpha/2}$ up to the boundary, provided the compatibility conditions (22.12) are satisfied at the initial manifold $t = 0$.

Finally, the existence of the solution is proved using the method of continuation with respect to a parameter $\lambda \in [0,1]$. We define a family of operators $L_\lambda = (1-\lambda)L^{(0)} + \lambda L^{(1)}$ and show that the set of λ for which the problem is solvable is both open and closed in $[0,1]$. This concludes the proof of the existence and uniqueness of the solution to the general boundary value problem (18.1)-(18.2).

References

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Note: Figure translations are in progress. See original paper for figures.