ON THE EQUATIONS OF THE DYNAMICS OF A BAROCLINIC OCEAN

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Abstract

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The formulation of problems in ocean dynamics has been studied in a number of investigations ($^{1-9}$). In this direction a number of significant results have been achieved. In recent years, in connection with the problem of interaction between the atmosphere and the ocean, interest in hydrodynamic theories of oceanic circulation has increased substantially.

Of special interest in this connection has been the use of the complete system of equations of ocean dynamics, taking baroclinic effects into account, for solving the problem of forecasting sea currents. In the present work the problem of ocean dynamics is considered and algorithms for its solution are formulated. The article considers gradient currents in an ocean of infinite extent in the (x, y) plane. The ocean bottom coincides with the plane $z = H$. We shall consider the linearized equations of ocean-current dynamics in the form

$$\begin{gathered}
\frac{\partial u}{\partial t} - lv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + lu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \\
\partial p/\partial z = g\rho, \\
\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0, \\
\partial T/\partial t + w\,\partial T/\partial z = 0, \\
\partial S/\partial t + w\,\partial S/\partial z = 0.
\end{gathered} \tag{1}$$

Here u, v, w are the components of the velocity vector $\mathbf{u}$; p is pressure; ρ is density; T is temperature; S is salinity; and l is the Coriolis parameter. To the system of equations (1) we append the equation of state in the form

$$\rho = f(T, S). \tag{2}$$

As boundary conditions on the free surface of the ocean $z = \zeta(x, y, t)$, we shall take the following:

$$w = \partial\zeta/\partial t. \tag{3}$$

We shall write this condition approximately for the equation $z = 0$. We transform equation (3), using the obvious relation between the pressure p at the surface $z = 0$, the atmospheric pressure p_0, which for simplicity we shall regard as constant, and the free surface ζ. Indeed, we have

$$p = p_0 - g\rho\zeta \quad \text{for } z = 0. \tag{4}$$

This means that condition (3) can be written in the form

$$\partial p/\partial t + g\rho w = 0 \quad \text{for } z = 0. \tag{5}$$

The boundary condition at the ocean bottom we set as follows:

$$w = 0 \quad \text{for } z = H. \tag{6}$$

We shall seek bounded solutions in the entire domain of definition of the solution, satisfying the initial data

$$u = u^0; \quad v = v^0; \quad T = T^0; \quad S = S^0 \quad \text{for } t = 0. \tag{7}$$

Thus, problem (1), (2), (5), (6), (7) has been posed completely. The problem formulated above can be somewhat simplified by passing from physical quantities to their deviations from the standard ones, which depend only on the coordinate z. Thus, if φ is some hydrophysical characteristic, then $\varphi = \overline{\varphi}(z) + \varphi'$. In addition, introduce the notation

$$d\overline{T}/dz = \Gamma_T, \qquad d\overline{S}/dz = \Gamma_S,$$

where Γ_T and Γ_S are functions of z. Then, on the basis of an order-of-magnitude analysis, we arrive at the following problem.

First consider the equation of state. Setting $\rho = \overline{\rho} + \rho'$, $p = \overline{p} + p'$, and $S = \overline{S} + S'$, to within small quantities of higher order, we shall have

$$\rho' = \frac{\partial f}{\partial \overline{T}} T' + \frac{\partial f}{\partial \overline{S}} S'. \tag{8}$$

Next consider the equations of heat and salinity inflow. To within small quantities of higher order they will have the form

$$\partial T'/\partial t + \Gamma_T w = 0,$$

$$\partial S'/\partial t + \Gamma_S w = 0. \tag{9}$$

Multiply the first equation of system (9) by $\partial f/\partial \overline{T}$, the second by $\partial f/\partial \overline{S}$, and add the results. Then we arrive at one equation for ρ',

$$\partial \rho'/\partial t + \Gamma w = 0, \tag{10}$$

where

$$\Gamma = \frac{\partial f}{\partial \overline{S}} \Gamma_S + \frac{\partial f}{\partial \overline{T}} \Gamma_T. \tag{11}$$

Let us note that the quantities $\dfrac{\partial f}{\partial \overline{S}} \Gamma_S > \Theta$ and $\dfrac{\partial f}{\partial \overline{T}} \Gamma_T > 0$.

Taking into account the considerations set forth above, we write the system of equations (1) in the form

$$\frac{\partial u}{\partial t} - lv = -\frac{1}{\overline{\rho}} \frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + lu = -\frac{1}{\overline{\rho}} \frac{\partial p}{\partial y},$$

$$\partial p/\partial z = g\rho,$$

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0,$$

$$\partial \rho/\partial t + \Gamma w = 0. \tag{12}$$

Here p and ρ are deviations from the standard values. The primes have been omitted for simplicity. As boundary conditions we take conditions (5), (6), which, for the deviations, have the form

$$\partial p/\partial t + g\overline{\rho} w = 0 \quad \text{for } z = 0, \tag{13}$$

$$w = 0 \quad \text{for } z = H. \tag{14}$$

We reduce the system of equations (12) to one equation for p. For this purpose we differentiate the last equation with respect to z,

$$\frac{\partial}{\partial z}\frac{1}{\Gamma}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z} w = 0. \tag{15}$$

Using the equation of continuity, we shall have

$$\frac{\partial}{\partial z}\frac{1}{\Gamma}\left(\frac{\partial \rho}{\partial t}\right) - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{16}$$

From equation (16) we eliminate ρ. Then we obtain

$$\frac{1}{g}\frac{\partial}{\partial z}\frac{1}{\Gamma}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{17}$$

Let us next consider the first two equations of system (12). By differentiation and elimination one can arrive at the relations

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} + l^2\right) u &= -\frac{l}{\rho}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial^2 t}{\partial x\, \partial t}, \\ \left(\frac{\partial^2}{\partial t^2} + l^2\right) v &= \frac{l}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial^2 t}{\partial y\, \partial t}. \end{aligned} \tag{18}$$

Act on equation (17) with the operator $\partial^2/\partial t^2 + l^2$ and eliminate u and v with the help of (18), under the assumption that l is a linear function of y. Then, to within small quantities, we obtain

$$\left[\left(\frac{\partial^2}{\partial t^2} + l^2\right)\frac{\bar{\rho}}{g}\frac{\partial}{\partial z}\frac{1}{\Gamma}\frac{\partial}{\partial z} + \Delta\right]\frac{\partial p}{\partial t} + \beta\frac{\partial p}{\partial t} = 0. \tag{19}$$

To obtain the boundary conditions, we use relations (13), (14) and eliminate w from these equalities with the aid of the equation for ρ. As a result we shall have

$$(\partial/\partial z - \Gamma/\bar{\rho})\, \partial p/\partial t = 0 \quad \text{for } z = 0; \tag{20}$$

$$\partial^2 p/\partial z\, \partial t = 0 \quad \text{for } z = H. \tag{21}$$

We proceed to the solution of the problem (19), (20), (21). For this purpose we investigate the spectral problem

$$\frac{d}{dz}\frac{1}{\Gamma}\frac{d\varphi}{dz} + \lambda\varphi = 0,$$

$$\frac{1}{\Gamma}\frac{d\varphi}{dz} - \frac{1}{\bar{\rho}}\varphi = 0 \quad \text{for } z = 0, \tag{22}$$

$$d\varphi/dz = 0 \quad \text{for } z = H.$$

It is not difficult to verify that problem (22) is self-adjoint in the sense of Lagrange, i.e., the following functional equality holds for functions g, h belonging to some class R:

$$(g, Lh) = (h, Lg), \tag{23}$$

where the scalar product is understood in the sense of the metric of Hilbert space. Problem (22) is self-adjoint; it determines a complete system of eigenfunctions $\varphi_n(z)$ and the corresponding system of real eigenvalues λ_n. Let us show that all the eigenvalues of problem (22) are positive. For this purpose it is necessary to establish the positive definiteness of the operator

$$L = -\frac{d}{dz}\frac{1}{\Gamma}\frac{d}{dz}$$

on the class of functions R, whose elements φ satisfy the boundary conditions from (22). The criterion for the positive definiteness of the operator L, as is known, has the form

$$J = (\varphi, L\varphi) > 0. \tag{24}$$

We write the functional J in explicit form:

$$J = -\int_0^H \varphi\,\frac{d}{dz}\frac{1}{\Gamma}\frac{d\varphi}{dz}\,dz. \tag{25}$$

With the help of integration by parts and taking into account the boundary conditions for the function φ, we arrive at the equality

$$J = \frac{1}{\bar{\rho}}\varphi^2(0) + \int_0^H \frac{1}{\Gamma}\left(\frac{d\varphi}{dz}\right)^2 dz > 0. \tag{26}$$

Here it is assumed that $\Gamma > 0$. Hence follows the positive definiteness of the operator L.

Since we now have a complete set of eigenfunctions of the operator $\{\varphi_n\}$ and the corresponding spectrum $\{\lambda_n\}$, with all $\lambda_n >$

> 0, then we can seek solutions of problem (19), (20), (21) in the form of the expansion

$$p = \sum_{m=0}^{\infty} p_m(x, y, t)\varphi_m(z), \tag{27}$$

where the functions $\varphi_n(z)$ are assumed to be orthonormal. We substitute (27) into (19), (20), (21) and multiply the result scalarly by $\varphi_n(z)$. Then we arrive at the system of equations

$$\left[\Delta - \frac{\lambda_n \bar{\rho}}{g}\left(\frac{\partial^2}{\partial t^2} + l^2\right)\right]\frac{\partial p_n}{\partial t} + \beta\frac{\partial p_n}{\partial x} = 0 \quad (n = 0, 1, 2, ...). \tag{28}$$

The correctness of the system of equations (28) for $\lambda_n > 0$ is evident.

We now turn to the formulation of the boundary conditions in the case when the sea is bounded by a closed cylindrical surface S. The natural condition in this case will be the condition of no normal flow, $u_{nv} = 0$ on S. With the aid of (18) one can obtain the boundary condition for P_n

$$\frac{\partial}{\partial v}\left(\frac{\partial p_n}{\partial t}\right) = l\,\frac{\partial p_n}{\partial s} \quad \text{on } S \quad (n = 0, 1, 2, ...). \tag{29}$$

Here v is the normal and s the tangential component of the unit vector.

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Note: Figure translations are in progress. See original paper for figures.