ON STRICTLY COSINGULAR OPERATORS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.88:513.83+517.948.35+517.948.5

MATHEMATICS

Yu. N. VLADIMIRSKII

ON STRICTLY COSINGULAR OPERATORS

(Presented by Academician P. S. Novikov on 31 VIII 1966)

All spaces under consideration are assumed to be Banach spaces, and all operators linear and bounded. An operator $A: X \to Y$ is called a $\Phi_+(\Phi_-)$-operator if $\operatorname{Im} A$ is closed and $\dim \operatorname{Ker} A < \infty$ ($\operatorname{codim} \operatorname{Im} A < \infty$). An operator $T: X \to Y$ is called **strictly singular** ([2]) if there does not exist an infinite-dimensional Banach space E with isomorphic embeddings $i_1: E \to X$, $i_2: E \to Y$ such that $Ti_1 = i_2$. An operator $T: X \to Y$ is called **strictly cosingular** ([3]) if there does not exist an infinite-dimensional Banach space E with quotient maps $h_1: X \to E$ and $h_2: Y \to E$ such that $h_2 T = h_1$. An operator $B: X \to Y$ is called a $\Phi_+(\Phi_-)$-**admissible perturbation** if $A + B$ is a $\Phi_+(\Phi_-)$-operator for every $\Phi_+(\Phi_-)$-operator $A: X \to Y$. It is known (see ([1,2])) that all strictly singular operators are Φ_+-admissible perturbations. In the present note it is established that all strictly cosingular operators are Φ_--admissible perturbations.

Following V. Pták, we shall use the notation $X \Subset Y$ to express that X is a closed subspace of Y. The space of all bounded linear operators acting from X to Y will be denoted by $L(X, Y)$.

Lemma 1. Let $T \in L(X, Y)$. Then:

a) T is strictly singular $\iff$ there does not exist an infinite-dimensional Banach space E with quotient maps $h_1: X^* \to E$ and $h_2: Y^* \to E$ such that $\operatorname{Ker} h_1$ is weakly closed and $h_1 T^* = h_2$.

b) T is strictly cosingular $\iff$ the restriction of T^* to any infinite-dimensional weakly closed subspace of Y^* is not an isomorphic embedding.

Lemma 2. Let $T \in L(X, Y)$. If the restriction of T^* to a weakly closed $Z_1 \Subset Y^*$ is not a Φ_+-operator, then for every $\varepsilon > 0$ there exists an infinite-dimensional weakly closed $Z_2 \Subset Z_1$ such that the restriction of T^* to Z_2 is completely continuous and has norm less than ε.

The proof is carried out analogously to the proof of Theorem 4.1 in ([1]). Biorthogonal sequences $\{y_k^*\}_1^\infty$ ($y_k^* \in Z_1$) and $\{y\}_1^\infty$ ($y_k \in Y$) are constructed such that

$$|y_k^*| = 1, \qquad |y_k| < 2^{2k-1}, \qquad |T^* y_k^*| < \varepsilon \cdot 2^{1-3k} \quad (k = 1, \dots, n, \dots).$$

Define the operator $A: Y^* \to X^*$ by the equality

$$Ay^* = \sum_{k=1}^{\infty} \langle y_k, y^* \rangle T^* y_k^*.$$

It is clear that A is completely continuous and $|A| < \varepsilon$. But $A = B^*$, where

$$Bx = \sum_{k=1}^{\infty} \langle x, T^* y_k^* \rangle y_k.$$

Therefore $\operatorname{Ker}(T^* - A) = \operatorname{Ker}(T - B)^*$ is weakly closed, and

$$Z_2 = Z_1 \cap \operatorname{Ker}(T^* - A)$$

is weakly closed. Since all $y_k^* \in Z_2$, it follows that $\dim Z_2 = \infty$. The restriction of T^* to Z_2 is completely continuous and has norm less than ε.

Theorem 1. Let $T \in L(X, Y)$. The following conditions are equivalent:

a) T is a strictly cosingular operator;

b) for every infinite-dimensional weakly closed $Z \subseteq Y^*$, the restriction of T^* to Z is not a Φ_+-operator;

c) for every infinite-dimensional weakly closed $Z_1 \subseteq Y^*$ there exists an infinite-dimensional weakly closed $Z_2 \subseteq Z_1$ such that the restriction of T^* to Z_2 is completely continuous.

Corollary 1. All strictly cosingular operators are Φ_--admissible perturbations. The set of all strictly cosingular operators from $L(X, Y)$ forms a closed subspace in $L(X, Y)$, which is a two-sided ideal if $X = Y$.

Proof. This follows from Proposition 1 in ([3]), Theorem 1, and Lemma 2.

From this corollary and Theorem 5.1 in ([1]) it follows that

Corollary 2. In the spaces l_p $(p \geqslant 1)$ and c_0, strictly cosingular operators are completely continuous.

The author expresses deep gratitude to A. S. Markus and D. A. Raikov for posing the problem and for their assistance in the work.

Moscow State
Pedagogical Institute
named after V. I. Lenin

Received
30 VIII 1966

References

[1] I. Ts. Gokhberg, A. S. Markus, I. A. Feldman, *Izv. Moldavsk. fil. AN SSSR*, No. 10 (76) (1960).
[2] T. Kato, *J. d' anal. math.*, 6 (1958).
[3] A. Pełczyński, *Bull. acad. polon. sci., Ser. sci. math., astr. et phys.*, 13, No. 1 (1965).

Note: Figure translations are in progress. See original paper for figures.