Oscillations of a pendulum with relay control

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Date: 1967-01-01T00:00:00+00:00

Abstract

The differential equation

$$\ddot{x} + a\dot{x} + f(x) = -u_0 \operatorname{sign}(\dot{x} - \varphi(x)),$$

is considered, where $a > 0$; $u_0 > 0$; $f(x)$ and $\varphi(x)$ are periodic and everywhere continuously differentiable functions that vanish at $x = 0$ and $x = +\pi$. This equation describes, in particular, the oscillations of a pendulum subjected to relay control.

In this paper, it is shown that for a sufficiently large value of the parameter u_0, the curve $\dot{x} = \varphi(x)$ is a sliding curve for the corresponding system of phase trajectories. In this situation, the author conducts a detailed analysis of the structure of the phase portrait of the system.

6 illustrations. 7 bibliography items.

Full Text

Preamble

This study investigates the dynamics of a nonlinear system described by the differential equation:

$$x'' + f(x) = u(x, x')$$

where the control function is defined as $u(x, x') = -u_0\text{sign}(x' - \phi(x))$ and the switching surface is given by $\phi(x) = \alpha + \phi_0(x)$. We assume $u_0 > 0$ and $\alpha > 0$. The function $f(x)$ is periodic such that $f(x) = -f(-x)$ and satisfies $f(0) = f(-\pi) = 0$. Furthermore, we assume $\phi_0(0) = \phi_0(-\pi) = 0$, with $f(x) > 0$ and $x\phi_0(x) < 0$ for $0 < |x| < \pi$. The control magnitude u_0 is assumed to be sufficiently large such that $u_0 > \max|\Phi(x)|$, where $\Phi(x) = f(x)+[\alpha+\phi_0'(x)]\phi(x)$.

The system can be rewritten as a first-order system:

$$\begin{aligned}
&\dot{x} = y \\
&\dot{y} = -\alpha y - f(x) - u_0\text{sign}(y - \phi(x))
\end{aligned}$$

This formulation allows for the analysis of the phase portrait in the (x, y) plane. We consider the behavior of trajectories relative to the switching line $y = \phi(x)$. Following the methods established in [**?**, **?**], we examine the existence and stability of equilibrium points and limit cycles. For the specific case where $f(x) = \sin x$, the system behavior is characterized by the interaction between the restoring force and the discontinuous control law.

Phase Space Analysis and Sliding Modes

The switching line $y = \phi(x)$ divides the phase plane into two regions. In the region where $y > \phi(x)$, the system follows $\dot{y} = -\alpha y - f(x) - u_0$, while for $y < \phi(x)$, it follows $\dot{y} = -\alpha y - f(x) + u_0$. A sliding mode exists on the segment of the switching line where the vector fields on both sides point toward the line. This condition is satisfied when:

$$\lim_{y \to \phi(x)^+} \dot{R} < 0 \quad \text{and} \quad \lim_{y \to \phi(x)^-} \dot{R} > 0$$

where $R(x, y) = y - \phi(x)$. Substituting the system equations, the sliding mode condition reduces to $u_0 > |\Phi(x)|$. Within the sliding domain, the motion is governed by the reduced-order equation $\dot{x} = \phi(x)$.

As shown in

, the phase trajectories demonstrate that for $0 < \alpha < -\min \phi_0(x)$, there exist stable equilibrium points. If the damping coefficient α exceeds this threshold, the qualitative nature of the phase portrait changes, potentially leading to the disappearance of certain singular points. We utilize the comparison method [**?**, **?**, **?**] to bound the trajectories. Specifically, we define comparison functions $Y(x)$ such that $Y(x) > \phi(x)$, allowing us to prove the convergence of trajectories to the sliding manifold or a specific limit cycle.

Stability and Equilibrium States

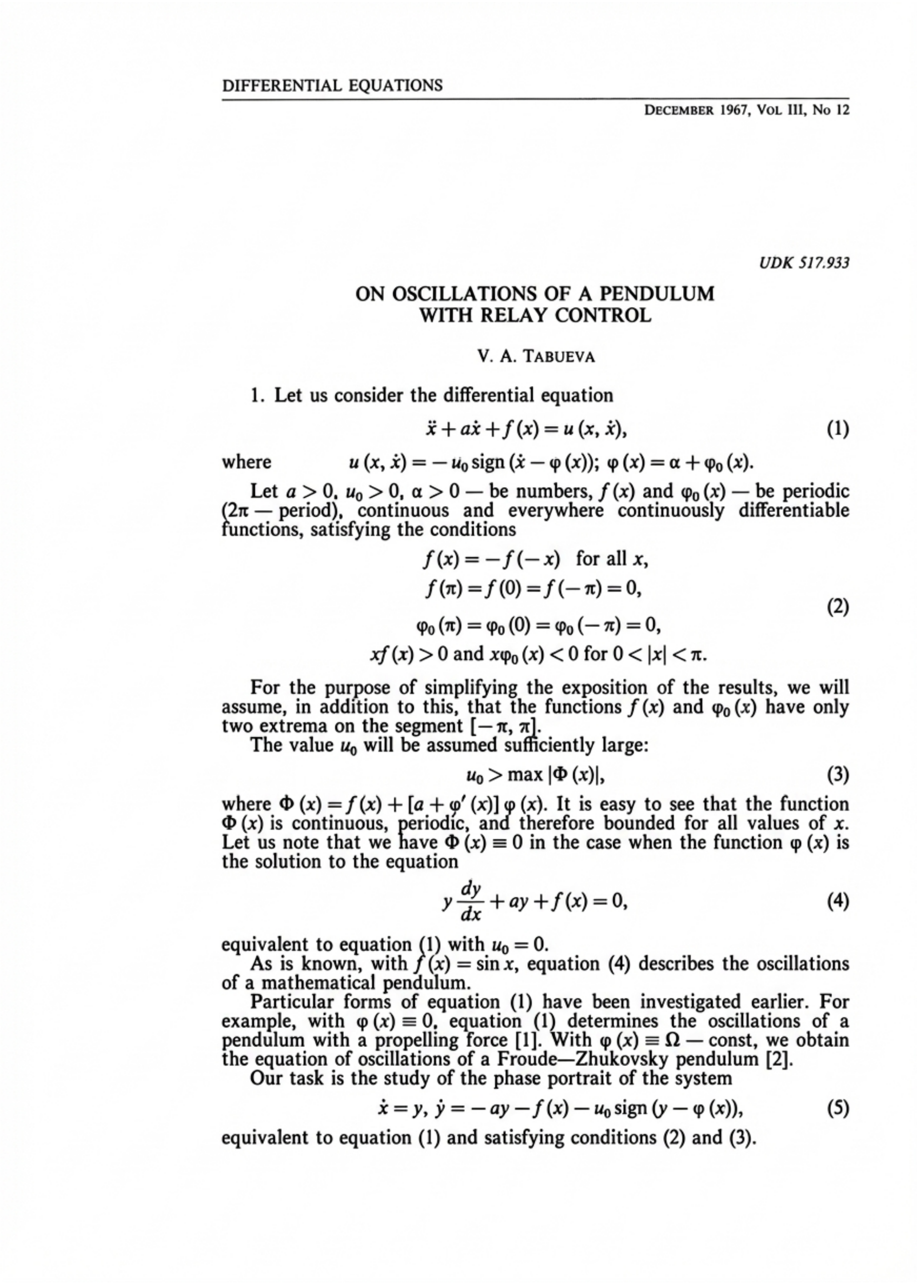
The equilibrium states of the system are located at the intersections of the nullclines. For the autonomous case, the points $(x_0, 0)$ are analyzed. Under the condition $u_0 > \max\{\max f(x), \max \Phi(x)\}$, the system exhibits robust stability. As illustrated in

, the trajectories starting from an initial state (x_0, y_0) eventually enter the sliding regime on $y = \phi(x)$ and converge to the origin or a periodic orbit.

In the interval $-\pi < x < \pi$, the behavior near the points $(x_1, 0)$ and $(x_2, 0)$ is critical. For $x = x_0 > 0$, the system trajectories are directed towards the switching line. If $\alpha > -\min \phi_0(x)$, the sliding motion is guaranteed to be stable.

and

provide a detailed visualization of the phase trajectories under varying parameters of u_0 and α. The global structure of the phase space is determined by



DIFFERENTIAL EQUATIONS

DECEMBER 1967, VOL III, NO 12

UDK 517.933

ON OSCILLATIONS OF A PENDULUM WITH RELAY CONTROL

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1. Let us consider the differential equation

$$\ddot{x} + a\dot{x} + f(x) = u(x, \dot{x}), \tag{1}$$

where $u(x, \dot{x}) = -u_0 \operatorname{sign}(\dot{x} - \varphi(x));\ \varphi(x) = \alpha + \varphi_0(x).$

Let $a > 0$, $u_0 > 0$, $\alpha > 0$ — be numbers, $f(x)$ and $\varphi_0(x)$ — be periodic (2π — period), continuous and everywhere continuously differentiable functions, satisfying the conditions

$$\begin{gathered} f(x) = -f(-x) \quad \text{for all } x, \\ f(\pi) = f(0) = f(-\pi) = 0, \\ \varphi_0(\pi) = \varphi_0(0) = \varphi_0(-\pi) = 0, \\ xf(x) > 0 \text{ and } x\varphi_0(x) < 0 \text{ for } 0 < |x| < \pi. \end{gathered} \tag{2}$$

For the purpose of simplifying the exposition of the results, we will assume, in addition to this, that the functions $f(x)$ and $\varphi_0(x)$ have only two extrema on the segment $[-\pi, \pi]$.

The value u_0 will be assumed sufficiently large:

$$u_0 > \max |\Phi(x)|, \tag{3}$$

where $\Phi(x) = f(x) + [a + \varphi'(x)]\,\varphi(x)$. It is easy to see that the function $\Phi(x)$ is continuous, periodic, and therefore bounded for all values of x. Let us note that we have $\Phi(x) \equiv 0$ in the case when the function $\varphi(x)$ is the solution to the equation

$$y\frac{dy}{dx} + ay + f(x) = 0, \tag{4}$$

equivalent to equation (1) with $u_0 = 0$.

As is known, with $f(x) = \sin x$, equation (4) describes the oscillations of a mathematical pendulum.

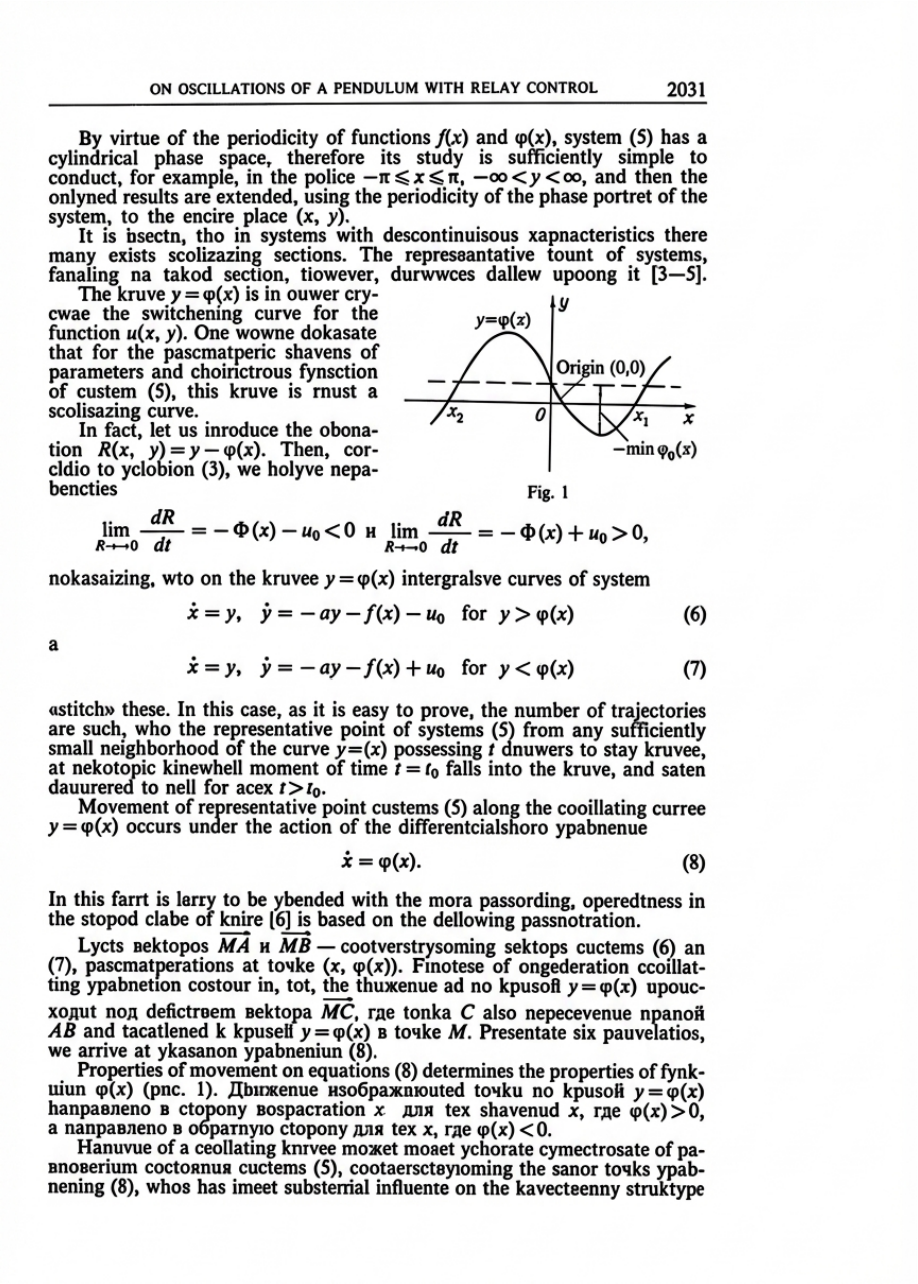
Particular forms of equation (1) have been investigated earlier. For example, with $\varphi(x) \equiv 0$, equation (1) determines the oscillations of a pendulum with a propelling force [1]. With $\varphi(x) \equiv \Omega$ — const, we obtain the equation of oscillations of a Froude—Zhukovsky pendulum [2].

Our task is the study of the phase portrait of the system

$$\dot{x} = y,\ \dot{y} = -ay - f(x) - u_0 \operatorname{sign}(y - \varphi(x)), \tag{5}$$

equivalent to equation (1) and satisfying conditions (2) and (3).

Figure 1: Figure 1



By virtue of the periodicity of functions $f(x)$ and $\varphi(x)$, system (5) has a cylindrical phase space, therefore its study is sufficiently simple to conduct, for example, in the police $-\pi \leqslant x \leqslant \pi$, $-\infty < y < \infty$, and then the onlyned results are extended, using the periodicity of the phase portret of the system, to the encire place (x, y).

It is bsectn, tho in systems with descontinuisous xapnacteristics there many exists scolizazing sections. The represaantative tount of systems, fanaling na takod section, tiowever, durwwces dallew upoong it [3—5].

The kruve $y=\varphi(x)$ is in ouwer crycwae the switchening curve for the function $u(x, y)$. One wowne dokasate that for the pascmatperic shavens of parameters and choirictrous fynsction of custem (5), this kruve is rnust a scolisazing curve.

In fact, let us inroduce the obonation $R(x, y)=y-\varphi(x)$. Then, corcldio to yclobion (3), we holyve nepabencties

Fig. 1

$$\lim_{R\to+0} \frac{dR}{dt} = -\Phi(x) - u_0 < 0 \text{ н } \lim_{R\to-0} \frac{dR}{dt} = -\Phi(x) + u_0 > 0,$$

nokasaizing, wto on the kruvee $y=\varphi(x)$ intergralsve curves of system

$$\dot{x} = y, \quad \dot{y} = -ay - f(x) - u_0 \quad \text{for} \quad y > \varphi(x) \tag{6}$$

a

$$\dot{x} = y, \quad \dot{y} = -ay - f(x) + u_0 \quad \text{for} \quad y < \varphi(x) \tag{7}$$

«stitch» these. In this case, as it is easy to prove, the number of trajectories are such, who the representative point of systems (5) from any sufficiently small neighborhood of the curve $y=(x)$ possessing t dnuwers to stay kruvee, at nekotopic kinewhell moment of time $t=t_0$ falls into the kruve, and saten dauurered to nell for acex $t>t_0$.

Movement of representative point custems (5) along the cooillating curree $y=\varphi(x)$ occurs under the action of the differentcialshoro ypabnenue

$$\dot{x} = \varphi(x). \tag{8}$$

In this farrt is larry to be ybended with the mora passording, operedtness in the stopod clabe of knire [6] is based on the dellowing passnotration.

Lycts вektopos $\overrightarrow{MA}$ и $\overrightarrow{MB}$ — cootverstrysoming sektops cuctems (6) an (7), pascmatperations at toчke $(x, \varphi(x))$. Finotese of ongederation ccoillatting ypabnetion costour in, tot, the thuжenue ad no kpusofi $y=\varphi(x)$ upoucxoдut noд defictrвem вektopa $\overrightarrow{MC}$, гдe tonka C also nepecevenue npanoй AB and tacatlened k kpuseй $y=\varphi(x)$ в toчke M. Presentate six pauvelatios, we arrive at ykasanon ypabneniun (8).

Properties of movement on equations (8) determines the properties of fynkuiun $\varphi(x)$ (pnc. 1). Дьıжenue изoбpaжnıouted toчku no kpusoй $y=\varphi(x)$ hanpaвлeno в ctopony вospacration x для tex shavenud x, гдe $\varphi(x)>0$, a nanpaвлeno в oбpatnyıo ctopony для tex x, гдe $\varphi(x)<0$.

Hanuvue of a ceollating knrvee moжet moaet ychorate cymectrosate of paвnoвerium coctoяnия cuctems (5), cootaersctвyıoming the sanor toчks ypabnening (8), whos has imeet substerrial influente on the kavecteenny struktype

Figure 2: Figure 2

Thus, only one possibility remains for the mutual arrangement of the curves $y=\varphi(x)$ and $y=Y(x)$, defined by condition (10) of the theorem.

The proven theorem states, in particular, the impossisbility of the ecsоs-existence of a limit cycle of the second kind for system (5), coinciding with a cycle of system (7).

3. Varieties of the phase portrait of system (5) significently depend on the type of fynction $\varphi(x)$, in particular on the belivinude of the parameter α.

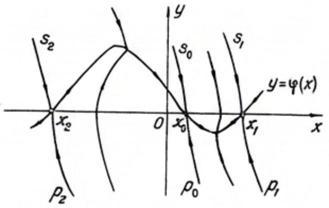


Fig. 2

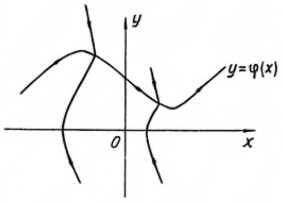


Fig. 3

For $0<\alpha<-\min\varphi_0(x)$ the arrangement of trajectories of system (5), subelnetrying conditions (2), upon fullinment of the ethio (9) is schematically depicted in pig. 2.

The painobecium statues of sustem (5) cornresponds to points $(x_2, 0)$, $(0, 0)$ axi $(x_1, 0)$, it $x=x_0>0$ и $x=x_2<0$ — zyres of the function $\varphi(x)$, bliwed to $x=0$, $x_1=x_2+2\pi$.

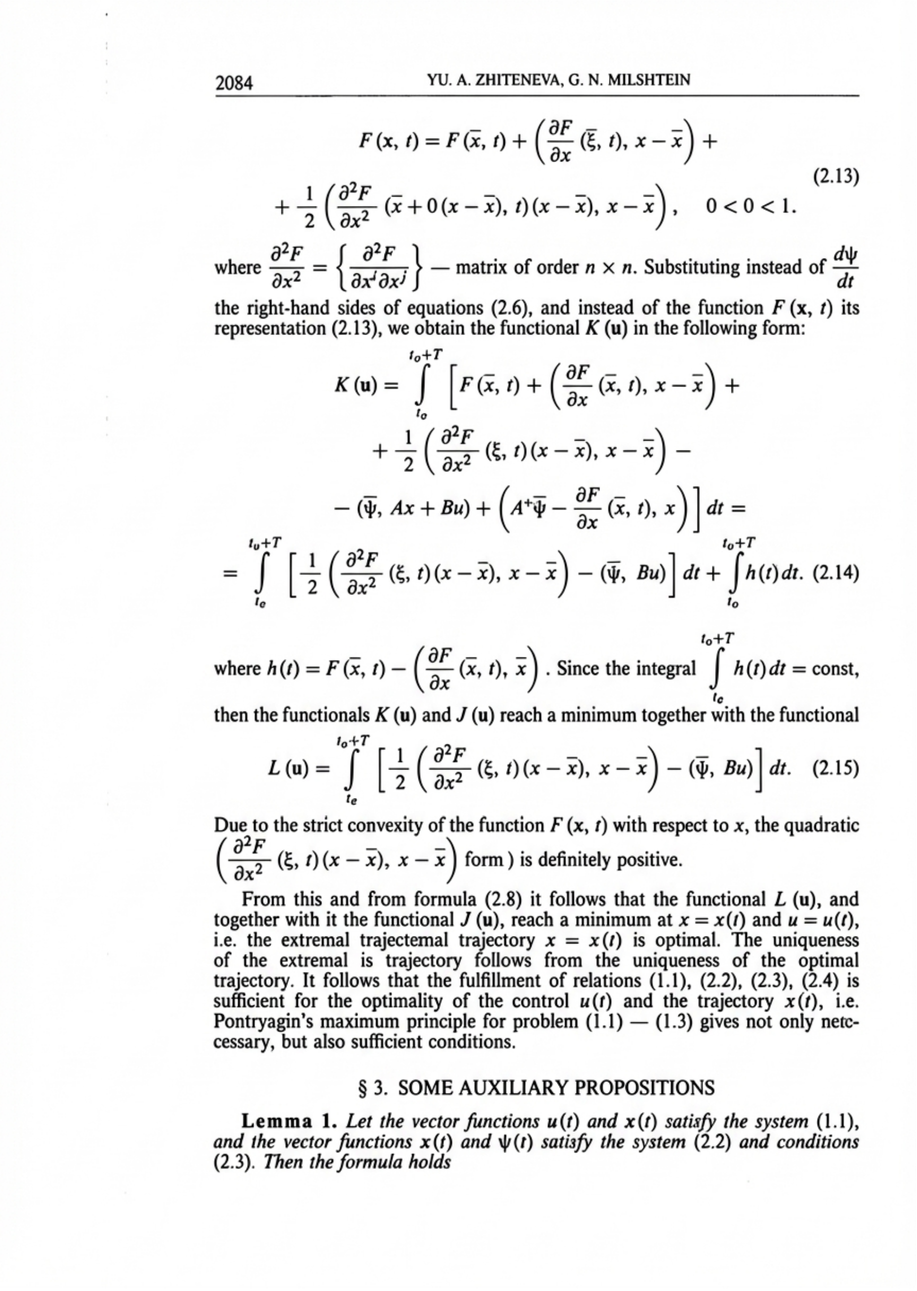
Tonkt $(x_0, 0)$ corrresponds, obviously, to a stobilar painobecium of the sustems. The arrangement of trajectories of system (5) near towkt $(x_0, 0)$ is sodofio to the parrangement trajectories in a doctatovnly small oquecc-nocth of singulars towk of type «stoible yzde». At that, trajectories s_0 and p_0 (pig. 2) prossoдит черес toчky $(x_0, 0)$ a nekotopый konewive moment of teone. The изображата toчki of sustemы (5), deliveing on octanoning trajectories of equernocth toчku $(x_0, 0)$, c pocton t обязateraiily попадает при nekotopом koneчном значении $t=t_0$ па kpивую ckolisikenия it and $t>t_0$ двигается по этой kpивой, примыкан при $t\to\infty$ к рассиmpиваемad toчke.

Tonks $(x_1\ 0)$ и $(x_3, 0)$ nodofiia to saddle singularsиtowиms. To search of these toчks npprыkaют тоnько vetыpe trajectories, две of hux and $t\to-\infty$, dpyree дve (s_1 и p_1 fon toиkt $(x_1, 0)$) pроходит черес особую toч-tnes at koneчнtime value of speme, touchasгa npяnoй $x=x_1$.

Bez much difficult, мoryt быть проесаsfiы рассуждения, nokasaisum-that, the yctoblish singularn toчka $(x_0, 0)$ umeet обnacts npriraжation, coonading with обnacno заnaura kpивой ckolisikenия $y=\varphi(x)$ for $x_2<x<x_1$. It is a noroitire, organuчennoиs trajectories s_1, p_1, s_2, p_2 (pig. 2). Thes one should wonly otmet smete the fact, that, since the движение изображающей toчku cuctemы (5) по kpивой $y=\varphi(x)$ происходит ac-coп по уравнению (8), то достаточно показать только попадание изобра-жающей точки из любой точки указанной полосы на криаую ckolisikenuп.

For $\alpha>-\min\varphi_0(x)$ the phasod portret sustem (5) npd yслotion (9) is schematically depictet in Fig. 3. Singulares towks cuctems (5) in thim cлyae, ovviously, do ne cymectвyut. The kruvia $y=\varphi(x)$ condineet a periodiectent perum in x, "impowend" relay ynpabrement: $-u_0\operatorname{sign}(y-\varphi(x))$, rде the beluvune u_0 gains according to nepaenctry (9).

Figure 3: Figure 3



$$F(\mathbf{x}, t) = F(\bar{x}, t) + \left(\frac{\partial F}{\partial x}(\bar{\xi}, t),\ x - \bar{x}\right) + \\ + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(\bar{x} + 0(x - \bar{x}),\ t)(x - \bar{x}),\ x - \bar{x}\right), \quad 0 < 0 < 1. \tag{2.13}$$

where $\frac{\partial^2 F}{\partial x^2} = \left\{\frac{\partial^2 F}{\partial x^i \partial x^j}\right\}$ — matrix of order $n \times n$. Substituting instead of $\frac{d\psi}{dt}$ the right-hand sides of equations (2.6), and instead of the function $F(\mathbf{x}, t)$ its representation (2.13), we obtain the functional $K(\mathbf{u})$ in the following form:

$$K(\mathbf{u}) = \int_{t_o}^{t_o+T} \left[F(\bar{x}, t) + \left(\frac{\partial F}{\partial x}(\bar{x}, t),\ x - \bar{x}\right) + \right. \\ + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(\xi, t)(x - \bar{x}),\ x - \bar{x}\right) - \\ \left. - (\bar{\psi},\ Ax + Bu) + \left(A^{+}\bar{\psi} - \frac{\partial F}{\partial x}(\bar{x}, t),\ x\right)\right] dt = \\ = \int_{t_o}^{t_o+T} \left[\frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(\xi, t)(x - \bar{x}),\ x - \bar{x}\right) - (\bar{\psi},\ Bu)\right] dt + \int_{t_o}^{t_o+T} h(t)\, dt. \tag{2.14}$$

where $h(t) = F(\bar{x}, t) - \left(\frac{\partial F}{\partial x}(\bar{x}, t),\ \bar{x}\right)$. Since the integral $\int_{t_o}^{t_o+T} h(t)\, dt = \text{const}$, then the functionals $K(\mathbf{u})$ and $J(\mathbf{u})$ reach a minimum together with the functional

$$L(\mathbf{u}) = \int_{t_o}^{t_o+T} \left[\frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(\xi, t)(x - \bar{x}),\ x - \bar{x}\right) - (\bar{\psi},\ Bu)\right] dt. \tag{2.15}$$

Due to the strict convexity of the function $F(\mathbf{x}, t)$ with respect to x, the quadratic $\left(\frac{\partial^2 F}{\partial x^2}(\xi, t)(x - \bar{x}),\ x - \bar{x}\right)$ form) is definitely positive.

From this and from formula (2.8) it follows that the functional $L(\mathbf{u})$, and together with it the functional $J(\mathbf{u})$, reach a minimum at $x = x(t)$ and $u = u(t)$, i.e. the extremal trajectemal trajectory $x = x(t)$ is optimal. The uniqueness of the extremal is trajectory follows from the uniqueness of the optimal trajectory. It follows that the fulfillment of relations (1.1), (2.2), (2.3), (2.4) is sufficient for the optimality of the control $u(t)$ and the trajectory $x(t)$, i.e. Pontryagin's maximum principle for problem (1.1) — (1.3) gives not only netcessary, but also sufficient conditions.

§ 3. SOME AUXILIARY PROPOSITIONS

Lemma 1. *Let the vector functions* $\mathbf{u}(t)$ *and* $\mathbf{x}(t)$ *satisfy the system* (1.1), *and the vector functions* $\mathbf{x}(t)$ *and* $\psi(t)$ *satisfy the system* (2.2) *and conditions* (2.3). *Then the formula holds*

Figure 4: Figure 4

the synthesis of these local behaviors, ensuring that for a wide range of initial conditions, the system tracks the desired sliding surface $\phi(x)$.

Conclusion

The analysis demonstrates that the discontinuous control law effectively forces the system state onto the surface $y = \phi(x)$. The existence of a stable sliding mode is contingent upon the control gain u_0 overcoming the effective nonlinearity $\Phi(x)$. The results presented in [FIGURE:5] and [FIGURE:6] confirm that the system achieves the desired regulation or tracking performance, even in the presence of nonlinear restoring forces $f(x)$. This approach remains valid for various functional forms of $\phi_0(x)$, provided the structural conditions outlined in the preamble are maintained.

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