A periodic boundary value problem for the differential equation $y^{(n)} + f(t, y, y^{'}, \dots, y^{(n-1)}) = 0$

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Abstract

The paper considers the conditions for the existence and uniqueness of the solution to the specified problem, while simultaneously providing estimates for both the solution and its derivatives. The main results are Theorems 1, 2, 3, and 4. The periodic boundary value problem for third- and fourth-order differential equations is investigated in detail. Bibliography: 20 items.

Full Text

Preamble

This work, published in 1967 (Volume III, No. 10), investigates the boundary value problem for the n-th order differential equation $y^{(n)} + f(t, y, \dots, y^{(n-1)}) = 0$. Building upon the foundational research of A. Ya. Khokhryakov [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?], we consider the general equation $y^{(n)} + f(t, y) = 0$ and its second-order counterpart $y'' + f(t, y, y') = 0$. The boundary conditions are defined as $a_{11}y(a) + a_{12}y'(a) = 0$ and $a_{21}y(\beta) + a_{22}y'(\beta) = 0$, following the frameworks established in [?] and [?].

Specifically, we examine the periodic boundary value problem:

$$y^{(n)} + f(t, y, \dots, y^{(\nu)}) = 0$$

$$y^{(k)}(a) - y^{(k)}(\beta) = 0, \quad (k = 0, \dots, \nu)$$

where f is a function of $t, y, \dots, y^{(\nu)}$. This study extends the results presented in [?] regarding the existence and uniqueness of solutions for equations of the form $L[y] = y^{(n)} + \sum g_k(t)y^{(k)} = 0$ under periodic conditions $y^{(k)}(a) - y^{(k)}(\beta) = 0$ for $k = 0, \dots, n-1$. We assume $g_k(t) = q_k$ and $a < \beta < T_0$. According to [?], the operator L maps the space of functions on $[a, \beta]$ into a corresponding function space, where the Green' s function $\Gamma(t, s)$ plays a critical role in establishing the existence of solutions.

1. Existence and Uniqueness Theorems

Consider the operator $L[y] = y^{(n)} + f(t, y, \dots, y^{(\nu)})$ in the region R, where $\nu < n$. We assume the function f satisfies conditions similar to those in [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]. Let L_1 and L_2 be linear operators such that the differential equation can be expressed as $y' + f(t, y) = 0$. We define a transformation $x = Ty$, where T is an $n \times n$ matrix, leading to the transformed system:

$$x' + B(t)x = 0, \quad x(a) - x(\beta) = 0$$

The matrix $B(t)$ is related to the coefficients $g_k(t)$ of the original equation. For the system to possess a unique solution, we require $\det B(t) = g_0(t) > 0$. Furthermore, we assume the coefficients $g_i(t)$ satisfy the conditions:

$$b_k(t) \leq g_k(t) \leq 0, \quad (k = 0, \dots, n-2)$$

where $b_k(t)$ are defined by the structural properties of the operator L.

For the third-order case ($n = 3$), the equation $y''' + g_2(t)y'' + g_1(t)y' + g_0(t)y = 0$ with periodic boundary conditions $y^{(k)}(a) - y^{(k)}(\beta) = 0$ ($k = 0, 1, 2$) is solvable if the coefficients satisfy:

$$\begin{aligned} b_0(t) &= g_0(t) - g_1(t) + g_2(t) - 1 \geq 0 \\ b_1(t) &= g_1(t) - 2g_2(t) + 3 < 0 \\ b_2(t) &= g_2(t) - 2g_0(t) > 0 \end{aligned}$$

Under these conditions, the Green' s function $\Gamma(t, s)$ exists for all $t, s \in [a, \beta]$.

2. Comparison and Monotonicity

Let $z(t)$ and $u(t)$ be functions in $C^1[a, \beta]$ such that $z \geq u$. We define the operator T such that $T(z - u) > 0$ for $t \in [a, \beta]$. If $y(t)$ is a solution to the boundary value problem (1.1), and there exist upper and lower solutions $z(t)$ and $u(t)$ satisfying:

$$\frac{dz}{dt} + A(t)z + f(t, z) - Q(t)z > 0$$

$$\frac{du}{dt} + A(t)u + f(t, u) - Q(t)u < 0$$

then the solution $y(t)$ is bounded by $u(t) \leq y(t) \leq z(t)$. This monotonicity property is essential for applying fixed-point theorems. Specifically, if the function f satisfies a Lipschitz-like condition:

$$f(t, y_1) - f(t, y_2) < -Q(t)(y_1 - y_2)$$

then the operator K defined by the Green' s function is a contraction, ensuring the uniqueness of the solution $x(t) = Kx(t)$ in the interval $[Tu, Tz]$.

3. Convergence and Error Estimates

Consider the system (3.1) where $B(t)$ is an $n \times n$ matrix. Let $B_1 = \min B(t)$ and $B_2 = \max B(t)$ be matrices in the class Λ^+. The Green' s functions $\Gamma_1(t,s)$ and $\Gamma_2(t,s)$ corresponding to these constant matrices provide bounds for the actual Green' s function $\Gamma(t,s)$ of the time-varying system:

$$\Gamma_2(t,s) > \Gamma(t,s) > \Gamma_1(t,s)$$

The existence of a solution $y(t)$ to the nonlinear problem (3.1) can be established if the norm of the operator satisfies $B_{11}L\epsilon < \epsilon$, where L is the Lipschitz constant of f and $\epsilon = (1, \dots, 1)$. The error between two approximate solutions $y(t)$ and $\bar{y}(t)$ is bounded by:

$$|y(t) - \bar{y}(t)| \leq \int_a^\beta |\Gamma(t,s)|L|y(s) - \bar{y}(s)|ds$$

This leads to the convergence criterion $A_1^{-1}L < 1$, where $A_1 = \min g_0(t)$.

4. Conclusion

The periodic boundary value problem for the n-th order equation $y^{(n)} + g(t)y + f(t,y) = 0$ possesses a unique solution if $g(t) > M(t) > 0$ and the nonlinear term f satisfies the growth condition $|f(t,y)| \leq M(t)|y| + \psi(t)$. The results presented here generalize several previous findings in the literature [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] and provide a robust framework for the qualitative analysis of higher-order differential equations with periodic constraints.

Note: Figure translations are in progress. See original paper for figures.

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