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Abstract

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ON THE FORMULATION OF THE ADIABATIC HYPOTHESIS IN AXIOMATIC FIELD THEORY

(Presented by Academician N. N. Bogolyubov on 9 II 1967)

In constructing quantum field theory starting from a Lagrangian formulation, an important role is played by the so-called adiabatic hypothesis. This hypothesis is essential for the construction of perturbation theory, since it is precisely what makes it possible to define the zeroth approximation of the theory—the free fields. However, along this path we are led to the necessity of distinguishing between "bare" and "physical" particles and to a number of other difficulties ([1]). This prompted the search for a way out in a formulation in which, from the very beginning, only physical asymptotic fields are introduced, and the S-matrix is defined axiomatically ([2,3]). But even in this case we have to deal with generalized functions and their products, with which is connected the freedom in choosing the continuation of the S-matrix beyond the energy surface ([4,5]).

This question is closely connected with the purely mathematical problem of the proper completion of the definition of certain integrals and of establishing the order of operations in expressions containing, in limiting form, zeros, poles, and δ-functions as factors. The adiabatic hypothesis of perturbation theory serves precisely as a prescription for such a completion of the definition. In the axiomatic formalism we must still formulate certain rules that will be equivalent to the adiabatic hypothesis.

2. A typical example of an expression requiring completion of the definition is the integral containing derivatives of the δ-function

$$\int dx_1\, dx_2\, (-K_{x_1})\delta(x_1-x_2)\varphi(x_1)\varphi(x_2); \quad -K_x = \partial^2/\partial x_0^2 - \partial^2/\partial x^2 + m^2. \; (1)$$

The kernel of this integral, $(-K_{x_1})\delta(x_1-x_2)$, is a generalized function whose meaning is revealed when integrating with test functions satisfying certain conditions. However, one usually has to deal with similar integrals also when the functions φ_1 and φ_2 do not satisfy all these requirements. In this situation it is desirable to have certain rules whose formal application would make it possible to obtain the correct answer.

Therefore—in the interests of methodological clarity—we shall consider in full detail a one-dimensional example that conveys all the features of the situation:

$$J = \lim_{\substack{b,\ \beta\to+\infty \\ a,\ \alpha\to-\infty}} J' = \lim_{\substack{b,\ \beta\to+\infty \\ a,\ \alpha\to-\infty}} \int_a^b dz_1 \int_\alpha^\beta dz_2\, \delta''(z_1 - z_2)\varphi_1(z_1)\varphi_2(z_2). \tag{2}$$

Acting quite naively, we may, in computing this integral, arrive at one of three results for J':

$$\int_a^b dz\, \varphi_1''(z)\varphi_2(z), \qquad \int_a^b dz\, \varphi_1'(z)\varphi_2'(z), \qquad \int_a^b dz\, \varphi_1(z)\varphi_2''(z), \tag{3}$$

which pass into one another after an equally naive integration by parts.

This procedure would be entirely legitimate if the functions φ_1 and φ_2 belonged to the class $C(p, q)$ with sufficiently large q [(2)], for in that case the boundary terms vanish. However, if φ_1, φ_2 are, for example, free solutions or Green' s functions of the Klein–Gordon equation, then the boundary terms of an integral of the type

$$\frac{1}{2}\left[\varphi_1'(z)\varphi_2(z) - \varphi_1(z)\varphi_2'(z)\right]_a^b \tag{4}$$

do not vanish. Indeed, the one-dimensional Klein–Gordon equation has a general solution of the form

$$\varphi(z) = a^{(+)}e^{imz} + a^{(-)}e^{-imz}, \tag{5}$$

and its causal Green' s function is

$$G(z) = \frac{i}{2m}e^{-im|z|}, \qquad \frac{\partial G}{\partial z} = \frac{\varepsilon(z)}{2}e^{-im|z|}. \tag{6}$$

Let us now consider the basic integral (2) in the third form (3), putting $\varphi_1(z) = G(z)$ and $\varphi_2(z) = \varphi(z)$:

$$I = \int_a^b dz\, G(x - z)\left(-\frac{\partial^2}{\partial z^2} - m^2\right)\varphi(z) = 0. \tag{7}$$

The equality to zero is ensured here by the fact that $\varphi(z)$ is a solution of the free equation. Performing integration by parts, we bring I to the form

$$I = \int_a^b dz \left\{ \left(-\frac{\partial^2}{\partial z^2} - m^2 \right) G(x-z) \right\} \varphi(z) + \left[-G \frac{\partial \varphi}{\partial z} + \frac{\partial G}{\partial z} \varphi \right]_a^b . \tag{8}$$

Now, if we were to discard the boundary term, we would obtain precisely the first form in (3), with $I_1 = -\varphi(x) \neq 0$. But in fact the boundary term (under the condition $x \in (a,b)$) is not equal to zero and, on the contrary, restores the validity of (7), turning I into zero. Thus, in the cases of interest for quantum field theory the three expressions (3) do not coincide and are equal to the original integral I only after they have been supplemented by boundary terms.

The possibility of mixing the various forms (3) increases especially upon passing to the p-representation, where all three forms look the same, $\int dk\, \varphi_1(k) k^2 \varphi_2(k)$. In the example considered there is obtained precisely the expression, well known from investigations of the asymptotic condition [(6)],

$$\frac{1}{m^2 - \omega^2 - i\varepsilon} (m^2 - \omega^2) \delta(m^2 - \omega^2). \tag{9}$$

This example teaches us that specialization of one of the forms (3) corresponds to specialization of the order of operations in (9), when one obtains either $\delta(m^2 - \omega^2)$, or 0.

Let us return to the original integral and carry out the integration more carefully. For this purpose, note that

$$\left\{ \frac{\partial^2}{\partial z_1^2} \delta(z_1 - z_2) \right\} \varphi(z_1) = \frac{\partial^2}{\partial z_1^2} \{\delta(z_1 - z_2)\varphi(z_1)\} - 2\frac{\partial}{\partial z_1} \{\delta(z_1 - z_2)\varphi'(z_1)\} + \delta(z_1 - z_2)\varphi''(z_1). \tag{10}$$

Performing now the integration of the total derivatives with respect to z_1, we find (we assume that always $a < b$ and $\alpha < \beta$), after symmetrization in z_1 and z_2,

$$\begin{aligned} J' = {} & \frac{1}{2} \int dz \, \{\varphi_1''(z)\varphi_2(z) + \varphi_1(z)\varphi_2''(z)\} + \frac{1}{2}\theta(b - \beta)[\varphi_1'\varphi_2 - \varphi_1\varphi_2']_\beta \\ & - \frac{1}{2}\theta(\beta - b)[\varphi_1'\varphi_2 - \varphi_1\varphi_2']_b + \frac{1}{2}\theta(a - \alpha)[\varphi_1'\varphi_2 - \varphi_1\varphi_2']_a \\ & - \frac{1}{2}\theta(\alpha - a)[\varphi_1'\varphi_2 - \varphi_2'\varphi_1]_\alpha - \frac{1}{2}\delta(\beta - b)\varphi_1(b)\varphi_2(b) \\ & - \frac{1}{2}\delta(a - \alpha)\varphi_1(a)\varphi_2(a) - \frac{1}{2}\delta(\beta - b)\varphi_1(\beta)\varphi_2(\beta) \\ & - \frac{1}{2}\delta(a - \alpha)\varphi_1(\alpha)\varphi_2(\alpha). \end{aligned} \tag{11}$$

It is clear from the last formula that the results of simplifying the integral J' depend essentially on the particular manner in which α, a and b, β tend to the infinite limit:

If $[\beta, \alpha] \subset (b, a)$, then

$$J = \frac{1}{2}\int dz\,(\varphi_1''\varphi_2 + \varphi_1\varphi_2'') + \frac{1}{2}[\varphi_1'\varphi_1 - \varphi_1\varphi_2']_a^b = \int dz\,\varphi_1''\varphi_2. \tag{12A}$$

If $[b, a] \subset (\beta, \alpha)$, then

$$J = \frac{1}{2}\int dz\,(\varphi_1''\varphi_2 + \varphi_1\varphi_2'') - \frac{1}{2}[\varphi_1'\varphi_2 - \varphi_1\varphi_2']_a^b = \int dz\,\varphi_1\varphi_2''. \tag{12B}$$

If one adopts the condition of "symmetric" passage to the limit $a = \alpha,\ b = \beta$, then

$$J' = \int dz\,\varphi_1''\varphi_2 + \frac{1}{2}[\varphi_1(z)\varphi_2'(z) - \varphi_1'(z)\varphi_2(z)]_a^b - \delta(b-\beta)\varphi_1(b)\varphi_2(b) - \delta(a-\alpha)\varphi_1(a)\varphi_2(a). \tag{12C}$$

We have directly convinced ourselves that, without specifying a concrete way in which the limits of integration tend to infinity, the integral J in general has no definite meaning.

In the usual presentation one invokes the adiabatic hypothesis at this point. Instead, we shall, by prescription, additionally define the integral J by the natural requirement of symmetric passage to the limit and at the same time discard the "extra terms" : the last two terms in the preceding formula, proportional to $\delta(0)$. Below we shall show that these terms owe their appearance to the "nonadiabatic" operation and, with more careful reasoning, turn into zero.

Up to now we have considered the integral J formally. We now turn to a rigorous definition of the meaning of expressions of this type, i.e. to the definition of the integral J by means of integration with test functions from an appropriate class. Specifically, we shall use the device of smoothing the boundaries by means of smooth functions with compact support. Introduce a new definition for J in the form:

$$J = \lim_{\substack{b,\beta\to\infty \\ a,\alpha\to-\infty}} \int_{-\infty}^{\infty} dz_1\,dz_2\,\delta''(z_1 - z_2)\,\varphi_1(z_1)\chi_1(z_1)\varphi_2(z_2)\chi_2(z_2), \tag{13}$$

where χ_1 and χ_2 are infinitely differentiable functions having the property ($\delta > 0$)

$$\chi_1(z) = \begin{cases} 1, & z \in [b-\delta, a+\delta], \\ 0, & z \in [b+\delta, a-\delta]; \end{cases} \qquad \chi_2(z) = \begin{cases} 1, & z \in [\beta-\delta, \alpha+\delta], \\ 0, & z \in [\beta+\delta, \alpha-\delta]. \end{cases} \tag{14}$$

Now (13) can be transformed to the form (10), where instead of φ_i the products $\varphi_i \chi_i$ appear, and, using property (14) of the functions χ_i, all boundary terms can be omitted. Then

$$J = \int_{-\infty}^{\infty} dz\, (\varphi_1 \chi_1)''(\varphi_2 \chi_2). \tag{15}$$

Performing integration by parts and omitting—again quite legitimately—the boundary terms, we bring J to the form:

$$J = \int_{-\infty}^{\infty} \varphi_1'' \varphi_2 \chi_1 \chi_2 \, dz + \int_{-\infty}^{\infty} dz\, \chi_1' (\varphi_1' \varphi_2 - \varphi_2' \varphi_1) \chi_2 - \int_{-\infty}^{\infty} dz\, \chi_1' \chi_2' \varphi_1 \varphi_2. \tag{16}$$

Let us now put $\chi_1 = \chi_2$ (in the spirit of the assumption of symmetry). Then the second term is written as

$$\frac{1}{2} \int_{-\infty}^{\infty} dz \left(\frac{d}{dz} \chi^2(z) \right) (\varphi_1' \varphi_2 - \varphi_2 \varphi_2').$$

Since

the function $\chi^2(z)$ has exactly the same properties as $\chi(z)$, while its derivative is different from zero only in two narrow regions $b-\delta < z < b+\delta$, $a-\delta < z < a+\delta$. Therefore, by the mean-value theorem,

$$\frac{1}{2} \int dz \left(\frac{\partial}{\partial z} \chi^2(z) \right) (\varphi_1' \varphi_2 - \varphi_1 \varphi_2') = \frac{1}{2} \left[\varphi_1' \varphi_2 - \varphi_1 \varphi_2' \right]_a^b. \tag{17}$$

Thus the second term in (16), as $\delta \to 0$, goes over precisely into the boundary term of formula (12C). The first term in (16) goes over into the main term of (12C); as for the last term in (16), it leads to "extra terms," no matter how we choose the limiting form of the function χ.

However, in (16) one may carry out symmetrization with respect to φ_1 and φ_2 (taking $\chi_1 = \chi_2$). Then the second term disappears, and we have:

$$J = \frac{1}{2} \int dz\, \chi^2(z) (\varphi_1'' \varphi_2 + \varphi_1 \varphi_2'') - \int dz\, \frac{\partial \chi}{\partial z} \frac{\partial \chi}{\partial z} \varphi_1(z) \varphi_2(z). \tag{18}$$

Let us now note that in this form there is already no boundary term, and therefore no necessity arises to let $\delta \to 0$. If now, in order to transform the last term in (18), we use the mean-value theorem,

$$\int dz \left(\frac{\partial \chi}{\partial z}\right)^2 \varphi_1(z)\varphi_2(z) = \varphi_1(\xi_1)\varphi_2(\xi_1) \int_{b-\delta}^{b+\delta} dz \left(\frac{\partial \chi}{\partial z}\right)^2 + \varphi_1(\xi_2)\varphi_2(\xi_2) \int_{a-\delta}^{a+\delta} dz \left(\frac{\partial \chi}{\partial z}\right)^2,$$

$$(\xi_1 \in (b+\delta, b-\delta), \quad \xi_2 = (a-\delta, a+\delta)), \tag{19}$$

then we shall see that the integrals in (19) are of order δ^{-1}, and therefore:

If, in carrying out the limiting transition $b, -a \to \infty$, we simultaneously let δ tend to infinity, caring only that this tending be such that $\delta/b, \delta/a \to 0$, then (19) tends to zero, i.e. the second term in (18) disappears.

This special way of passing to the limit contains the idea of the adiabatic character and serves as a justification of the prescription proposed above for discarding "extra terms" in the symmetric form of the integral.

Thus, a suitable ("adiabatic") limiting process leads to the result

$$\int_{-\infty}^{\infty} dz_1\, dz_2\, \varphi_1(z_1)\delta''(z_1 - z_2)\varphi_2(z_2) = \frac{1}{2}\int_{-\infty}^{\infty} dz\, (\varphi_1''(z)\varphi_2(z) + \varphi_1(z)\varphi_2''(z)). \tag{20}$$

With the aid of the same prescription, formulas are obtained for arbitrary derivatives of the δ-function.

The formula obtained, (20), and its analogues contain a formal recipe for handling integrals with derivatives of the δ-function. The justification of such a calculation is provided by the usual procedure of considering similar integrals with the aid of smoothed functions; however, when it is necessary to compute complicated expressions containing repeated application of such a device, the proposed recipe makes it possible to arrive automatically at the correct result.

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Note: Figure translations are in progress. See original paper for figures.