THE CAUCHY PROBLEM FOR A FIRST-ORDER-IN-TIME EQUATION WITH VARIABLE COEFFICIENTS

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Abstract

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THE CAUCHY PROBLEM FOR A FIRST-ORDER-IN-TIME EQUATION WITH VARIABLE COEFFICIENTS

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In the present paper we consider the Cauchy problem for the equation

$$\mathcal{L}_{tx}u = \frac{\partial u}{\partial t} + \sum_{k=0}^{n}(-i)^{k+1}a_k(t,x)\frac{\partial^k u}{\partial x^k} = 0 \tag{1}$$

with the initial conditions

$$u\big|_{t=t_0} = \varphi(x) \tag{2}$$

in the finite strip in t, $0 \le t_0 \le t \le T,\ |x| < \infty$.

It is assumed that the coefficients of equation (1) satisfy the following conditions: the functions $a_k(t,x)$ $(k = 0,1,\dots,n)$ are sufficiently smooth in the strip $D:$ $0 \le t \le T,\ |x| < \infty$ (more precisely, see below); $\operatorname{Im} a_n(t,x) = 0$, $a_n(t,x) > 0$ (or $a_n(t,x) < 0$), $n \ge 2$; $\lim a_k(t,x) = b_k = \text{const}$ $(k = 0,1,\dots,n)$, $b_n \ne 0$, $\operatorname{Im} b_k = 0$ $(k = 1,2,\dots,n)$ (or $\operatorname{Im} b_n = \dots = \operatorname{Im} b_{p+1} = 0$, $\operatorname{Im} b_p > 0$ for some even $p,\ 2 \le p < n$), i.e. the "limiting" equation

$$\frac{\partial u}{\partial t} + \sum_{k=0}^{n}(-i)^{k+1}b_k\frac{\partial^k u}{\partial x^k} = 0 \tag{3}$$

is well posed in the sense of I. G. Petrovskii.

Previously, cases were considered in which the coefficients of equation (1) are constant (^{1,2}), or depend only on t (^{1}), or depend only on x (^{3}), or the case in which equation (1) is parabolic (^{4}). In the present paper equation (1) is not parabolic. Moreover, equation (1) with "frozen" coefficients may fail to be well posed in the sense of I. G. Petrovskii.

1°. Let us first consider the case $n \geq 3$. We shall assume that the coefficients of equation (1) satisfy the following condition:

a) the functions $\partial^l a_k(t,x)/\partial x^l$ $(l = 0, 1, \dots, k+2$ for $k = n-1, n;$ $l = 0, 1, 2$ for $k = 0, 1, \dots, n-2)$, $\partial^{l+1} a_k(t,x)/\partial x^l \partial t$ $(l = 0, 1, 2;$ $k = n-1, n)$ are continuous in D and satisfy the inequalities

$$\left| \frac{\partial^{l+r}\left[a_k(t,x) - b_k\right]}{\partial x^l\, \partial t^r} \right| \leq c(1+|x|)^{(l-n-k-3-\varepsilon)/(n-1)}, \tag{4}$$

where $l = 0, 1, \dots, k+2,$ $r = 0$ for $k = n-1, n;$ $l = 0, 1, 2,$ $r = 1$ for $k = n-1, n;$ $l = 0, 1, 2,$ $r = 0$ for $k = 0, 1, \dots, n-2;$ ε is some positive number; the constant $c > 0$, here and throughout the subsequent estimates, does not depend on the arguments of the functions.

We shall assume, without loss of generality, that $b_n = 1,$ $b_{n-1} = 0$. (This can always be achieved by the substitution $u(t,x) = \tilde{u}(\tilde{t},x)\exp\{-ib_{n-1}x/nb_n\}$, $\tilde{t} = b_n t$.)

The solution of problem (1)–(2) is sought in the form

$$u(t, t_0, x) = \int G(t, t_0, x, \xi)\varphi(\xi)\, d\xi, \tag{5}$$

where, here and everywhere below, the integral denotes integration over the whole line; the function $G(t, t_0, x, \xi)$ is the solution of the equation

$$G(t, t_0, x, \xi) = G_0(t, t_0, x, \xi) - \int_{t_0}^{t} d\tau \int G(t, \tau, x, y)\mathcal{L}_{\tau y} G_0(\tau, t_0, y, \xi)\, dy; \tag{6}$$

$G_0(t, t_0, x, \xi)$ is the principal part of the function $G(t, t_0, x, \xi)$ as $t \to t_0$ and is represented explicitly as

$$\begin{aligned} G_0(t, t_0, x, \xi) = a_n^{(n-1)/2n}(t_0, x)\, a_n^{(-n-1)/2n}(t_0, \xi)\, \Gamma\left(t - t_0, \int_\xi^x a_n^{-1/n}(t_0, \eta)\, d\eta\right) \\ \times \exp\left\{-\frac{1}{n}\int_\xi^x a_{n-1}(t_0, \eta) a_n^{-1}(t_0, \eta)\, d\eta\right\}; \end{aligned} \tag{7}$$

$\Gamma(t, x)$ is the Green’ s function for equation (3), i.e.

$$\Gamma(t, x) = \frac{1}{2\pi}\int \exp\left\{it\sum_{k=0}^{n} b_{ks}^{k} + isx\right\} ds. \tag{8}$$

Equation (6) is solved by the ordinary method of successive approximations. It is proved that the function $G(t, t_0, x, \xi)$ is continuous in the domain $0 \leq t_0 < t \leq T,\ |x| < \infty,\ |\xi| < \infty$, together with its derivatives with respect to x up to order n and first order with respect to t, and satisfies the estimates

$$\left|\frac{\partial^{r G}(t, t_0, x, \xi)}{\partial x^r}\right| \leq \frac{c(1+|x|)^{(2r+2-n)/2(n-1)}(1+|\xi|)^{(n+2+\alpha_r)/2(n-1)}}{(t-t_0)^{\beta_r}}$$

$$(r = 0, 1, \dots, n), \tag{9}$$

where $\alpha_r = 0$ for $r = 0, 1, \dots, n-2$; $\alpha_{n-1} = 2$; $\alpha_n = 4$; $\beta_r = \max\{(1+r)/n,\ (1+2r)/2(n-1)\}$.

The proof is based on estimates of the Green' s function of equation (3):

$$\left|\frac{\partial^r \Gamma(t, x)}{\partial x^r}\right| \leq \frac{c}{t^{(1+r)/n}}\left(1 + \frac{|x|}{t^{1/n}}\right)^{(2r+2-n)/2(n-1)} \qquad (r = 0, 1, \dots),$$

which follow from the work of M. V. Fedoryuk (^{5}). From the properties of the function $G(t, t_0, x, \xi)$ there follows the theorem on the existence of a solution of problem (1)—(2).

Theorem 1. Let the coefficients of equation (1) satisfy condition a), and let the initial function $\varphi(x)$ satisfy the condition:

b) the functions $\varphi^{(l)}(x)$ $(l = 0, 1, 2)$ are continuous on the whole line and satisfy the estimate

$$|\varphi^{(l)}(x)| \leq c(1+|x|)^{(2l-3n-4-\varepsilon)/2(n-1)} \quad (l = 0, 1, 2)$$

for some $\varepsilon > 0$.

Then the function (5) is a classical solution of problem (1)—(2).

2°. In order that the coefficients of the equation adjoint to equation (1) satisfy condition a), it is necessary to require that the coefficients of equation (1), in addition to condition a), satisfy the condition:

c) the functions $\partial^l a_k(t, x)/\partial x^l$ $(l = 0, 1, \dots, k+2;\ k = 0, 1, \dots, n-2)$ are continuous in D and satisfy inequalities (4) for $l = 0, 1, \dots, k+2;\ r = 0;\ k = 0, 1, \dots, n-2$.

It is proved that, under conditions a) and c), the function $G(t, t_0, x, \xi)$, with respect to the variables t_0, ξ, satisfies the equation adjoint to equation (1). With the aid of the existence of a solution of the Cauchy problem for the equation adjoint to equation (1), the uniqueness theorem for the solution of problem (1)—(2) is proved.

Theorem 2. *Let the coefficients of equation (1) satisfy conditions a), c). Then the solution of problem (1)–(2) is unique in the class*

$$|u(t,t_0,x)| \leq c/(t-t_0)^{1-\varepsilon}(1+|x|)^{(n-4+\varepsilon)/2(n-1)}.$$

The continuous dependence of the solution of problem (1)–(2) on the initial data, as well as the uniform well-posedness of problem (1)–(2) with respect to t_0, follow directly from formula (5).

Similarly, the Cauchy problem is considered for the nonhomogeneous equation

$$\mathcal{L}_{tx}u = f(t,x), \qquad u|_{t=t_0} = 0,$$

whose solution is represented in the form

$$u(t,t_0,x) = \int_{t_0}^{t} d\tau \int G(t,\tau,x,\xi) f(\tau,\xi)\, d\xi.$$

3°. The case $n=2$ is considered analogously, but under additional smoothness assumptions on the coefficients of equation (1). This is due to the fact that the singularity of the Green function for equation (3) as $t \to 0$ for $n=2$ is substantially stronger than for $n \geq 3$. Therefore, in addition to condition a), it is required that the functions $\partial^{l+r}a_2(t,x)/\partial x^l \partial t^r$ ($r=1,2$; $l+r \leq 4$) be continuous in D and satisfy estimate (4). In this case the function $G_0(t,t_0,x,\xi)$ has the form

$$G_0(t,t_0,x,\xi) = a_2^{1/4}(t_0,x)a_2^{-3/4}(t_0,\xi)\,\Gamma\left(t-t_0, \int_\xi^x a_2^{-1/2}(t_0,\eta)\,d\eta\right) \times$$

$$\times \exp\left\{-\frac{i}{2}\int_\xi^x a_1(t_0,\eta)a_2^{-1}(t_0,\eta)\,d\eta + \frac{i}{4}\int_\xi^x a_2^{-3/2}(t_0,\eta)\frac{\partial a_2(t_0,\eta)}{\partial t_0}\int_\xi^\eta a_2^{-1/2}(t_0,\beta)\,d\beta\,d\eta\right\}.$$

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Note: Figure translations are in progress. See original paper for figures.

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