ON \(Q\)-SPACES

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Abstract

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MATHEMATICS

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ON Q-SPACES

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E. Hewitt, in the paper ([1]), defined the class of Q-spaces. Most often these spaces are called functionally closed.

Definition. A completely regular space X is called a Q-space (functionally closed) if every maximal countably centered system of zero-sets has a nonempty intersection. Here a system $\xi = \{Z_\alpha\}$ is called countably centered if every countable subsystem of this system of sets has a nonempty intersection. A set $A \subseteq X$ is called a zero-set if on the space X there exists a continuous function such that the set A is the complete preimage of zero.

Theorem 1. *A perfect image of a normal countably paracompact Q-space X is always normal, countably paracompact, and is a Q-space.*

Proof. Let $f : X \to Y$ be a perfect mapping, and let X be a normal countably paracompact Q-space. Then the space Y is normal and countably paracompact (for this it is sufficient only that the mapping f be closed). We shall prove that Y is a Q-space. Let $\xi = \{Z_\alpha\}$ be an arbitrary maximal countably centered system of zero-sets of the space Y. This means that in the space Y there exist continuous functions $\varphi_\alpha(y)$ such that $Z_\alpha = \varphi_\alpha^{-1}(0)$. Then $\psi_\alpha(x) = \varphi_\alpha(fx)$ is a continuous function on the space X, and

$$\psi_\alpha^{-1}(0) = f^{-1}(\varphi_\alpha^{-1}(0)) = f^{-1} Z_\alpha,$$

therefore the system

$$\eta = f^{-1}\xi = \{f^{-1} Z_\alpha\}$$

is countably centered and consists of zero-sets of the space X. The system η, generally speaking, is not a maximal countably centered system of zero-sets. Complete it to a maximal countably centered system $\hat{\eta}$ of zero-sets of the space X. We shall prove that in our case the system obtained, η, will be countably centered.

Consider an arbitrary countable subsystem $\hat{\eta}_0 = \{\Phi_{\lambda_i}\} \subseteq \hat{\eta}$ of the system $\hat{\eta}$. Without loss of generality we may assume that

$$\Phi_{\lambda_1} \supseteq \Phi_{\lambda_2} \supseteq \cdots \supseteq \Phi_{\lambda_i} \supseteq \cdots,$$

since the system $\hat{\eta}$, together with any two of its elements, also contains their intersection. Consider the system

$$f\hat{\eta}_0 = \{f\Phi_{\lambda_i}\}$$

of subsets closed in Y. Since for arbitrary i, α one has

$$\Phi_{\lambda_i} \cap f^{-1}Z_\alpha \neq \Lambda,$$

it follows that

$$f\Phi_{\lambda_i} \cap Z_\alpha \neq \Lambda$$

for all i, α. We shall now prove that

$$\bigcap_{i=1}^{\infty} f\Phi_{\lambda_i} \neq \Lambda.$$

Suppose the contrary; let

$$\bigcap_{i=1}^{\infty} f\Phi_{\lambda_i} = \Lambda.$$

Then, by virtue of the normality and countable paracompactness of the space Y (see ([2])), there exist neighborhoods $Of\Phi_{\lambda_i}$ of the sets $f\Phi_{\lambda_i}$ for which

$$\bigcap_{i=1}^{\infty} Of\Phi_{\lambda_i} = \Lambda.$$

Further, by Urysohn' s lemma, there exist zero-sets Z_i of the space Y such that

$$f\Phi_{\lambda_i} \subseteq Z_i \subseteq Of\Phi_{\lambda_i}.$$

Since

$$f\Phi_{\lambda_i} \cap Z_\alpha \neq \Lambda$$

for all i, α, the zero-set $Z_i \supseteq f\Phi_{\lambda_i}$,

system $\xi = \{Z_\alpha\}$ is a maximal countably centered system of zero-sets, then $Z_i \in \xi$, $i = 1, 2, \ldots$. Since $Z_i \subset Of\Phi_{\lambda_i}$, $\bigcap_{i=1}^{\infty} Of\Phi_{\lambda_i} = \Lambda$, it follows that $\bigcap_{i=1}^{\infty} Z_i = \Lambda$, and we have arrived at a contradiction with the countable centeredness of the system ξ. Thus, necessarily $\bigcap_{i=1}^{\infty} f\Phi_{\lambda_i} \neq \Lambda$. Let $y_0 \in \bigcap_{i=1}^{\infty} f\Phi_{\lambda_i}$ be an arbitrary point.

Consider the countable system $\{f^{-1}y_0 \cap \Phi_{\lambda_i}\}_{i=1}^{\infty}$ of nonempty closed subsets of the bicompactum $f^{-1}y_0$. Since $\Phi_{\lambda_1} \supseteq \Phi_{\lambda_2} \supseteq \cdots \supseteq \Phi_{\lambda_i} \supseteq \cdots$, we have

$$\bigcap_{i=1}^{\infty} (f^{-1}y_0 \cap \Phi_{\lambda_i}) \neq \Lambda,$$

and hence the system $\hat{\eta}$ is countably centered. Thus, we have extended our countably centered system $\eta = f^{-1}\xi = \{f^{-1}Z_\alpha\}$ of zero-sets of the space X to a maximal, also countably centered, system $\hat{\eta}$ of zero-sets. In view of the fact that X is a Q-space, necessarily

$$\bigcap_{\Phi_\lambda \in \hat{\eta}} \Phi_\lambda \neq \Lambda.$$

Then $\bigcap_\alpha f^{-1} Z_\alpha \neq \Lambda$, and consequently also

$$\bigcap_\alpha Z_\alpha \neq \Lambda,$$

i.e. Y is a Q-space. Theorem 1 is proved.

Remark 1. In the proof of Theorem 1 we used only the countable compactness of $f^{-1} y_0$.

Remark 2. This theorem is a partial answer to a question of V. I. Ponomarev from (4): is functional closedness of a normal space preserved under a perfect mapping? In Frolik' s paper (3) a theorem is formulated which gives a complete answer to the question, but the proof of this theorem contains a gap.

By the same method one also proves the following known result (see (4)).

Proposition (V. I. Ponomarev). *An open perfect image of a normal Q-space is always a Q-space.*

In proving this proposition the following assertion is used:

Lemma 1. *The image $f\Phi$ of a zero-set Φ in a space X under an open perfect mapping $f : X \to Y$ is a zero-set of the space Y.*

Theorem 2. *Let X be a normal countably paracompact space. In order that X be a Q-space, it is necessary and sufficient that every maximal countably centered system of closed (not necessarily zero-) sets have nonempty intersection.*

Proof. Necessity. Let $\xi = \{F_\lambda\}$ be a maximal countably centered system of closed sets. Denote by ξ_0 the collection of all zero-sets belonging to the system ξ. Clearly, ξ_0 is countably centered. We show that it is maximal.

Let Z be a zero-set which meets all elements of the system $\xi_0 = \{Z_\lambda\}$. If $Z \notin \xi = \{F_\lambda\}$, then there is $F_{\lambda_0} \in \xi$, $F_{\lambda_0} \cap Z = \Lambda$. By the normality of X there exists a zero-set Z_0 such that $Z_0 \supseteq F_{\lambda_0}$, $Z_0 \cap Z = \Lambda$. But since ξ is a maximal countably centered system, $Z_0 \in \xi$; consequently, $Z_0 \in \xi_0$, and then $Z_0 \cap Z \neq \Lambda$ (by the choice of Z). Thus $Z \in \xi$, and consequently $Z \in \xi_0$. Since X is a Q-space, we have

$$\bigcap_{Z_\lambda \in \xi_0} Z_\lambda \neq \Lambda.$$

Furthermore, any closed subset $F \subseteq X$, in particular $F_\lambda \in \xi$, is representable as an intersection of zero-sets, i.e. $F_\lambda = \bigcap_\alpha Z_\alpha^\lambda$; $Z_\alpha^\lambda \in \xi_0$. Then

$$\bigcap_\lambda F_\lambda = \bigcap_\lambda \left(\bigcap_\alpha Z_\alpha^\lambda \right) = \bigcap_{Z \in \xi_0},$$

which proves the necessity.

Sufficiency. Let $\xi_0 = \{Z_\lambda\}$ be a maximal countably centered-

system of zero-sets of the space X. Extend it to a maximal simply centered system $\xi = \{F_\lambda\}$ of closed sets. We shall prove the countable centeredness of the system ξ. Let $\{F_{\lambda_i}\}$ be some countable subsystem of the system ξ. We must prove that

$$\bigcap_{i=1}^{\infty} F_{\lambda_i} \neq \Lambda.$$

Suppose the contrary, let

$$\bigcap_{i=1}^{\infty} F_{\lambda_i} = \Lambda.$$

Then, by normality and countable paracompactness (see ([2])) of the space X, there exist neighborhoods OF_{λ_i} of the closed sets F_{λ_i}, for which also

$$\bigcap_{i=1}^{\infty} OF_{\lambda_i} = \Lambda.$$

Moreover, there exist zero-sets Z_i such that

$$F_{\lambda_i} \subseteq Z_i \subseteq OF_{\lambda_i}.$$

Since $F_{\lambda_i} \in \xi$, it follows that $Z_i \in \xi$, and consequently $Z_i \in \xi_0$. Therefore, on the one hand,

$$\bigcap_{i=1}^{\infty} Z_i \subset \bigcap_{i=1}^{\infty} OF_{\lambda_i} \neq \Lambda,$$

and on the other hand, $Z_i \in \xi_0$, i.e.

$$\bigcap_{i=1}^{\infty} Z_i \neq \Lambda$$

by the countable centeredness of the system ξ_0. We have obtained a contradiction. Hence the system $\xi = \{F_\lambda\}$ is countably centered. Then

$$\bigcap_{\lambda} F_\lambda \neq \Lambda,$$

and consequently, a fortiori,

$$\bigcap_{Z \in \xi_0} Z \neq \Lambda,$$

i.e. the space X is a Q-space, which proves sufficiency, and with it the whole theorem.

Remark. In proving necessity we did not use the countable paracompactness of X.

Theorem 3. *Let $f : X \to Y$ be a continuous mapping of a normal space X onto a hereditarily finally compact space Y, such that the preimage $f^{-1}y$ of every point $y \in Y$ is a Q-space. Then X is a Q-space.*

Proof. Denote by $\xi = \{Z_\lambda\}$ some arbitrary maximal countably centered system of zero-sets of the space X. We must prove that

$$\bigcap_\lambda Z_\lambda \neq \Lambda.$$

Consider the system

$$f\xi = \{fZ_\lambda\}$$

of sets in Y. The system $f\xi$ is also countably centered. We shall show that

$$\bigcap_\lambda fZ_\lambda \neq \Lambda.$$

By the final compactness of the space Y we shall have

$$\bigcap_\lambda [fZ_\lambda]_Y \neq \Lambda,$$

where $[fZ_\lambda]_Y$ is the closure of the set fZ_λ in Y. Moreover, observe that

$$\bigcap_\lambda [fZ_\lambda]_Y$$

consists of not more than one point.

Let

$$\bigcap_\lambda [fZ_\lambda] = y_0.$$

Fix some index λ_0 and consider the system

$$\eta_{\lambda_0} = \{fZ_{\lambda_0} \cap fZ_\lambda\}.$$

This system is countably centered and consists of sets belonging to the finally compact space

$$fZ_{\lambda_0} \subseteq Y.$$

Therefore

$$A = \bigcap_\lambda ([fZ_{\lambda_0} \cap fZ_\lambda]_{fZ_{\lambda_0}}) \neq \Lambda.$$

But

$$A \subseteq \bigcap_\lambda [fZ_{\lambda_0} \cap fZ_\lambda]_Y \subseteq \bigcap_\lambda ([fZ_{\lambda_0}]_Y \cap [fZ_\lambda]_Y) \subseteq \bigcap_\lambda [fZ_\lambda]_Y = y_0.$$

It follows (by the arbitrariness of λ_0) that

$$y_0 = \bigcap_\lambda fZ_\lambda.$$

Now consider the system

$$\{f^{-1}y_0 \cap Z_\lambda\} = \xi_{y_0}.$$

This system is countably centered and consists of zero-sets in $f^{-1}y_0$. We shall prove its maximality.

Let $Z_0 \subseteq f^{-1}y_0$ be a zero-set of some function φ defined on $f^{-1}y_0$. By normality of the space X (Urysohn' s theorem) there exists an extension $\widetilde{\varphi}$ of the function φ to the whole space X.

Consider the zero-set $Z(\widetilde{\varphi})$ of the function $\widetilde{\varphi}$. We have

$$Z(\widetilde{\varphi}) \cap f^{-1}y_0 = Z_0.$$

Since y_0 is a zero-set in Y, $f^{-1}y_0$ is a zero-set in X. Therefore Z_0 is a zero-set in all of X as the intersection of the zero-sets $f^{-1}y_0$ and $Z(\widetilde{\varphi})$. Let

$$Z_0 \cap (f^{-1}y_0 \cap Z_\lambda) \neq \Lambda$$

for every λ. Then

$$Z_0 \cap Z_\lambda \neq \Lambda$$

for all λ, i.e. $Z_0 \in \xi = \{Z_\lambda\}$, and then $Z_0 \in \{f^{-1}y_0 \cap Z_\lambda\}$, $Z_0 \subseteq$

$\subseteq f^{-1}y_0$, whereby the maximality of the system ξ_{y_0} is proved. Thus, ξ_{y_0} is a maximal countably centered system of zero-sets of the subspace $f^{-1}y_0$. From the fact that $f^{-1}y_0$ is a Q-space, it follows that

$$\bigcap_\lambda (f^{-1}y_0 \cap Z_\lambda) \neq \Lambda,$$

and then all the more

$$\bigcap_\lambda Z_\lambda \neq \Lambda,$$

which proves the theorem.

Remark. Theorem 3 strengthens B. A. Pasynkov' s theorem from ([5]). In B. A. Pasynkov' s theorem the space Y has a countable base.

Theorem 4. *Let $f : X \to Y$ be a closed continuous mapping of a normal space X onto a Q-space Y. If every point $y \in Y$ has type G_δ in Y and its inverse image $f^{-1}y$ is a Q-space, then X is also a Q-space.*

The proof of Theorem 4 is simple, and we omit it.

Theorem 5. *Let $f : X \to Y$ be a closed continuous mapping of a countably paracompact normal space X onto a Q-space Y, under which the inverse image $f^{-1}y$ of every point $y \in Y$ is functionally closed. Then the space X is also functionally closed.*

Proof. Since the space X is normal and countably paracompact, by virtue of Theorem 2 it suffices for us to prove that every maximal countably centered system of closed (not necessarily zero-) sets has a nonempty intersection. Let $\xi = \{F_\lambda\}$ be an arbitrary maximal countably centered system of closed sets of the space X. Consider the system $f\xi = \{fF_\lambda\}$ of closed subsets of Y. The system $f\xi$ is a maximal countably centered system of closed sets in Y. Since Y is functionally closed, we have

$$\bigcap_\lambda fF_\lambda \neq \Lambda.$$

Let

$$y_0 \in \bigcap_\lambda fF_\lambda.$$

Then $f^{-1}y_0 \cap F_\lambda \neq \lambda$ for all λ. Hence it follows that $f^{-1}y_0 \in \xi$ by the maximality of the system ξ. Therefore the system

$$\{f^{-1}y_0 \cap F_\lambda\} = \xi_{y_0}$$

is a countably centered system of closed sets in $f^{-1}y_0$. It is easy to prove that ξ_{y_0} is maximal with respect to these properties. Since $f^{-1}y_0$ is functionally closed, we have

$$\bigcap_\lambda (f^{-1}y_0 \cap F_\lambda) \neq \Lambda,$$

and then all the more

$$\bigcap_\lambda F_\lambda \neq \Lambda.$$

Theorem 5 is proved.

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