ON APPROXIMATION BY ELEMENTS OF CONVEX SETS

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Abstract

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ON APPROXIMATION BY ELEMENTS OF CONVEX SETS

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In the paper (^{1}) A. L. Garkavi established duality relations for approximation by elements of convex sets in real Banach spaces. G. Sh. Rubinshtein (^{2}) investigated more general dual problems and obtained, on the basis of duality relations, a criterion for an element of best approximation (in a real space). Another criterion, whose derivation did not explicitly rely on duality relations, was given by A. L. Garkavi (^{3}) (see also ($^{4-6}$)). We also point to the important work of E. G. Gol' shtein (^{7}), in which the convex set was specified in a certain special way encompassing the most important cases for concrete applications.

In the present article we first of all give an extension of the relations from (^{1}) and of the criterion from (^{2}) to the complex case. Despite the extreme simplicity of this extension, it proves very useful for a number of problems in the theory of analytic functions. Such a problem is, for example, the well-known Walsh problem (see (^{8}), p. 442) on approximation by uniformly bounded analytic functions. Next, the connection is shown between the criterion from (^{2}) (in the form extended to the complex case) and the criterion from (^{3}). These theorems are refined for the case when the convex set is finite-dimensional. This refinement extends to the case of approximation by elements of convex sets the Singer criterion (^{9}), known for subspaces. We also extend to the case of approximation by elements of convex sets Garkavi' s theorems (^{10}) on de la Vallée Poussin alternation and show their connection with Singer-type criteria.

Let X be a real or complex Banach space (moreover, the completeness of X is often inessential), $G \subset X$ a convex set. Let $y \notin \overline{G}$. Denote by $G^*(y)$ the collection of all linear functionals f in X^* satisfying the condition:

$$\inf_{x\in G} \operatorname{Re} f(x-y) \geq 1.$$

By $G_0^*(y)$ we denote the set of all $f \in X^*$ for which

$$\inf_{x\in G} \operatorname{Re} f(x-y) > 0.$$

Theorem 1. The following relations hold:

$$\inf_{x\in G}\|x-y\| = \max_{f\in G^*(y)}\left(\frac{1}{\|f\|}\right) = \max_{\substack{f\in G_0^*(y)\\ \|f\|=1}}\inf_{x\in G}\operatorname{Re} f(x-y). \tag{1}$$

(We write max and min instead of sup and inf where sup and inf are attained.)

Theorem 2. In order that an element $x_0 \in G$ be an element of best approximation to y, i.e. that

$$\|x_0-y\| = \inf\|x-y\|, \quad x\in G,$$

it is necessary and sufficient that there exist a functional $f_0 \in X^*$, $\|f_0\| = 1$, for which the following conditions are satisfied:

$$\operatorname{Re} f_0(x_0-y) = \|x_0-y\|, \qquad \operatorname{Re} f_0(x_0-x)\le 0, \quad x\in G. \tag{2}$$

Moreover, the functional f_0 may be taken to be one and the same for all elements of G giving the best approximation to y.

For the case when X is real, Theorem 1 is Garkavi' s duality relation ((1), Theorem 1). The proof of (1) in the complex case is carried out in the same way as in the real case, and we omit it. Theorem 2 in the real case was indicated in (2). Let us outline its proof. On the one hand, for f_0 realizing the max in the last part of (1), it is easy to verify that relations (2) hold. On the other hand, if relations (2) hold, then for every $x \in G$

$$\|x_0-y\| = \operatorname{Re} f_0(x_0-y) \le \operatorname{Re} f_0(x_0-y) + \operatorname{Re} f_0(x-x_0) \le \|x-y\|.$$

We now show the connection of the criterion contained in Theorem 2 with Garkavi' s criterion (3):

Theorem 3. *In order that $x_0 \in G$ be an element of best approximation to $y \in \overline{G}$, it is necessary and sufficient that, for every $x \in G$, $x \ne x_0$, there exist a functional f^x, which is an extreme point of the unit sphere S^* of the space X^*, such that*

$$\operatorname{Re} f^x(x_0-y) = \|x_0-y\|, \qquad \operatorname{Re} f^x(x_0-x)\le 0. \tag{3}$$

Proof. Sufficiency is proved as in Theorem 2. Let us dwell on the proof of the necessity of conditions (3). Introduce the notation:

$$\mathfrak{M}(x_0) = \{f;\ f\in S^*,\ \operatorname{Re} f(x_0-y) = \|x_0-y\|\},$$

$$Q(x) = \{f;\ f\in S^*,\ \operatorname{Re} f(x_0-x)\le 0\}, \quad B(x) = \mathfrak{M}(x_0)\cap Q(x).$$

Here $x \in G$ is fixed. It is obvious that the sets $\mathfrak{M}(x_0)$, $Q(x)$, $B(x)$ are bicompact in the topology $\sigma(X^*, X)$ and convex; moreover, the set $B(x)$ is nonempty, as follows from Theorem 2. Suppose first that there exists a functional $F \in B(x)$ such that $\operatorname{Re} F(x_0 - x) < 0$. By the Krein-Milman theorem, F is a limit point of the set of all possible convex combinations $\sum \lambda_j F_j$, where the F_j are extreme points of $\mathfrak{M}(x_0)$, and therefore, as is easy to see, extreme points of S^*. Consequently, there is such a convex combination $\sum \lambda_j F_j$ that $\sum \lambda_j \operatorname{Re} F_j(x_0 - x) < 0$. But then for at least one extreme point F_{j_0} it must be that $\operatorname{Re} F_{j_0}(x_0 - x) < 0$. Since, in addition, $F_{j_0} \in \mathfrak{M}(x_0)$, we may take $f^x = F_{j_0}$.

Now suppose that $\operatorname{Re} f(x_0 - x) = 0$, $f \in B(x)$. By the Krein-Milman theorem, $B(x)$ has an extreme point F. We shall show that now F is an extreme point of S^*. Indeed, if $F = \lambda F_1 + (1 - \lambda) F_2$, $0 < \lambda < 1$, $F_1, F_2 \in S^*$, then F_1 and F_2 belong to $\mathfrak{M}(x_0)$. Moreover, if, for example, $\operatorname{Re} F_1(x_0 - x) > 0$, then $\operatorname{Re} F_2(x_0 - x) < 0$ and, consequently, in $B(x)$ there would be found a functional F_2 for which $\operatorname{Re} F_2(x_0 - x) < 0$, which contradicts the supposition. Thus, $\operatorname{Re} F_1(x_0 - x) = \operatorname{Re} F_2(x_0 - x) = 0$, and, consequently, F_1 and F_2 belong to $B(x)$. Hence F cannot be an extreme point of $B(x)$—a contradiction. We can now put $f^x = F$. The theorem is proved.

Consider the case of a finite-dimensional convex set.

Theorem 4. *Let $G \subset X$ be a convex set of dimension n and $y \in \overline{G}$. In order that $x_0 \in G$ be an element of best approximation to y, it is necessary and sufficient that there exist r extreme points $\varphi_1, \dots, \varphi_r$ of the unit sphere S^* ($0 < r \le n + 1$ in the real case and $0 < r \le 2n + 1$ in the complex case) and numbers $\lambda_1 > 0, \dots, \lambda_r > 0$, $\sum_1^r \lambda_j = 1$, such that*

$$\sum_{j=1}^{r} \lambda_j \operatorname{Re} \varphi_j(x_0 - y) = \|x_0 - y\|, \qquad \sum_{1}^{r} \lambda_j \operatorname{Re} \varphi_j(x_0 - x) \le 0, \quad x \in G. \quad (4)$$

Proof. We shall carry out the argument for the real case, since the complex case introduces no additional difficulties.

Without loss of generality, assume that $0 \in G$. Otherwise, taking $x^* \in G$, $x^* \neq 0$, we would consider the problem of approximating the element

$y - x^*$ by elements of the set $G - x^*$, and the establishment of (4) for this problem would lead immediately to the same relations in the original situation. Let L denote the subspace of least dimension spanned by G and y. The dimension of L is not greater than $n + 1$ (and not less than n).

By Theorem 2, there exists a functional $\varphi \in L^*$, $\|\varphi\| = 1$, for which the relations hold:

$$\varphi(x_0 - y) = \|x_0 - y\|, \qquad \varphi(x_0 - x) \le 0, \quad x \in G.$$

Since φ lies on the boundary of the unit sphere of L^*, by Carathéodory's theorem (^{11}), there exist r, $0 < r \le n + 1$, extreme points of the unit sphere $\psi_1, \dots, \psi_r$

such that

$$\varphi = \sum_1^r \lambda_j \psi_j, \qquad \lambda_j > 0, \qquad \sum_1^r \lambda_j = 1.$$

Every functional that is an extreme point of the unit sphere in L^* can be extended to all of X in such a way as to be an extreme point of the sphere S^*. This fact was first noted in (^{9}), but its proof in that paper was incorrect; a complete proof may be found in (^{3}). Let the functional φ_j be an extension of ψ_j to all of X and serve as an extreme point for S^*. The combination

$$\sum_1^r \lambda_j \varphi_j$$

satisfies all the requirements of the theorem. The sufficiency of the conditions of our theorem is contained in Theorem 2. The proof is complete.

The theorem proved is an extension of Singer' s theorem (^{9}). As a corollary we derive from it a theorem on purification, generalizing Harkavy' s result (^{10}), and we emphasize the connection existing between criteria of Singer type and purification theorems.

Theorem 5. *In the situation of Theorem 4, the extreme points $\varphi_1, \dots, \varphi_r$ of the sphere S^* are such that*

$$\min_{x\in G} \|y - x\| = \min_{x\in G} \max_{f\in S^*} |f(y-x)| = \min_{x\in G} \max_{\varphi_1,\dots,\varphi_r} |\varphi_j(y-x)|. \tag{5}$$

Conversely, if for extreme points $\varphi_1, \dots, \varphi_r$ of the sphere S^* the relations (5) hold, then for some convex combination of them the conditions (4) are satisfied.

Proof. Consider the compact set Q, consisting of the points $\varphi_1, \dots, \varphi_r$, and the space $C(Q)$ of functions continuous on it, with the usual norm. In this space the problem is posed of approximating the continuous function $y(\varphi) = \varphi(y)$, $\varphi \in Q$, by the functions $x(\varphi) = \varphi(x)$, $\varphi \in Q$, which form a convex set when the element x runs through the set G. Let $\lambda_1, \dots, \lambda_r$ be the numbers determined in Theorem 4. For functions $z(\varphi) \in C(Q)$ consider the linear functional $l = \sum_1^r \lambda_j \operatorname{Re} z(\varphi_j)$. Obviously,

$$\|l\|_{C(Q)} \leq 1.$$

Since, moreover,

$$l[y(\varphi) - x_0(\varphi)] = \sum_{j=1}^r \lambda_j \operatorname{Re}[y(\varphi_j) - x_0(\varphi_j)] =$$

$$= \sum_{j=1}^r \lambda_j \operatorname{Re} \varphi_j(y - x_0) = \|y - x_0\| \geq \|y(\varphi) - x_0(\varphi)\|_{C(Q)},$$

it follows that $\|l\|_{C(Q)} = 1$, $\|y - x_0\| = \|y(\varphi) - x_0(\varphi)\|_{C(Q)}$. Further, for any $x(\varphi)$, $x \in G$, we have:

$$\operatorname{Re} l(x_0(\varphi) - x(\varphi)) = \sum_1^r \lambda_j \operatorname{Re} \varphi_j(x_0 - x) \leq 0.$$

Theorem 2 now shows that $x_0(\varphi)$ gives the best approximation to $y(\varphi)$, and since $\|y - x_0\| = \|y(\varphi) - x_0(\varphi)\|_{C(Q)}$, the proof-

The proof of the first assertion of Theorem 5 is complete. The second assertion is proved analogously. For the case of a set G of arbitrary dimension the following result holds, generalizing Theorem 2 of the paper by Garkavi [10] to the case of convex sets. We shall call a functional $f \in S^*$ maximal for an element $z \neq 0$ if $f(z) = \|z\|$.

Theorem 6. *Suppose that for $y \in X$ there exists an element $x_0 \in G$ of best approximation to y, and let Γ be the set of extreme points of the sphere S^* that are maximal for the element $x_0 - y$. Then*

$$\|y - x_0\| = \min_{x \in G} \sup_{f \in \Gamma} |f(x - y)|. \tag{6}$$

Proof. Let Q be the closure of Γ in the topology $\sigma(X^*, X)$. $Q \subset S^*$ and is bicompact in $\sigma(X^*, X)$. Clearly, in (6) one may replace Γ by Q. Let f_0 be the functional mentioned in Theorem 2. By the Krein–Milman theorem, there exists a positive Radon measure μ, $\int d\mu = 1$, concentrated on the set $\mathcal{E}$, which is the closure of the set of extreme points of the sphere S^*, such that for all $x \in X$ we have

$$f_0(x) = \int_{\mathcal{E}} \varphi(x)\, d\mu(\varphi).$$

It is easy to show that every point of growth of the measure μ belongs to Q. If we now consider the approximation in $C(Q)$ of the function $y(\varphi) = \varphi(y)$ by the functions $x(\varphi) = \varphi(x)$, $x \in G$, then the functional

$$l(z(\varphi)) = \operatorname{Re} \int_Q z(\varphi)\, d\mu = \operatorname{Re} \int_Q \varphi(z)\, d\mu(\varphi) = \operatorname{Re} f_0(z)$$

leads to the proof of the theorem in exactly the same way as was done in Theorem 5.

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Note: Figure translations are in progress. See original paper for figures.

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