ON THE MEAN VALUES OF THE SUPPORT FUNCTION

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Abstract

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1. Let G be a convex body in n-dimensional Euclidean space; $h(x,\nu)$ the distance from the point $x \in G$ to the supporting plane to G with outward normal ν; Ω the unit sphere (the set of all ν); χ_n its area. Define the mean values of the function $h(x,\nu)$

$$h_k(x) = \left[\frac{1}{\chi_n}\int_\Omega h^{-k}(x,\nu)\,d\nu\right]^{-1/k}, \qquad k \neq 0, \tag{1}$$

$$h_0(x) = \exp\frac{1}{\chi_n}\int_\Omega \ln h(x,\nu)\,d\nu. \tag{2}$$

These functions h_k deserve attention not only because interesting geometric problems are connected with them, but also because they majorize, up to a factor, the solutions $u(x)$ of second-order equations of a fairly general form under $u|_{\partial G} = 0$ ([1,2]). In ([2]), for linear equations the majorants are given in the form $Hh_k(x)$ with one or another k, $0 \leq k \leq n$, where H is expressed in terms of certain norms of the coefficients (and of the solution itself, if uniqueness conditions are not satisfied). Further, in ([3]) it is proved that the majorants with h_0 and h_n are sharp. Thus, for example, the function h_0 admits a definition through solutions of elliptic linear second-order equations: $h_0(s) = \sup|u(x)|$, where the supremum is taken over the solutions (with $u|_{\partial G} = 0$) of all equations for which $H = 1$.

2. Let us list some of the simplest properties of the function h_k.

(2.1) If $G \subset G'$, $G \neq G'$, then for every k and every $x \in G$

$$h_k(x) < h'_k(x).$$

(2.2) If $k > l$, then $h_k(x) < h_l(x)$, except in the case when G is a ball and x is its center. (This follows from the well-known property of means; see, for example, ([7]).)

(2.3) If $k > -1$, then everywhere inside G, $d^2 h_k < 0$, while for $k < -1$, $d^2 h_k > 0$. For the proof we note that

$$h(x, \nu) = h(x_0, \nu) + (x_0 - x)\nu.$$

Substituting this in (1) or (2), it is easy to find $d^2 h_k$ and to verify the asserted statement. In addition, we find that $h_{-1}(x) = \text{const}$.

(2.4) As is known, if from the point x in each direction ν one lays off a segment of length $r(x, \nu) = h^{-1}(x, \nu)$, then one obtains the convex body G^x, polar to G with respect to the point x (with respect to the unit ball with center x). Therefore, for $k \neq 0$,

$$h_k(x) = \left[\frac{1}{\chi_n} \int_\Omega r^k(x, \nu)\, d\nu\right]^{-1/k}. \tag{3}$$

Hence it is easy to conclude that if k is an integer, $1 \le k \le n$, then

$$h_k(x) = \tau_k^{1/k} \hat{V}_k^{-1/k}(x), \tag{4}$$

where τ_k is the volume of the k-dimensional unit ball, and $\hat{V}_k(x)$ is the mean value of the k-dimensional volumes of the sections of the body G^x by k-dimensional planes passing through x; in particular, $\hat{V}_n(x)$ is the volume of G^x. (The mean is understood here and below as the arithmetic mean in the sense of the natural measure in the set of ⋯

set of planes of the given dimension passing through a fixed point.)

What has been said makes it possible, among other things, to find $h_n(x)$ when the polar body G^x is simply determined. For example, the body polar to an ellipsoid is an ellipsoid. Its volume is found from elementary geometric considerations. Thus we find that for an ellipsoid

$$h_n(x) = (a_1, \dots, a_n)^{1/n} (1 - \rho^2(x))^{(n+1)/2n},$$

where a_i are the semiaxes, and $\rho(x)$ is the ratio of the distance from the center to x to the radius in the same direction.

3. Let us indicate estimates for $H_k \neq \max h_k$ in terms of the following quantities: V_m–the greatest of the volumes of the m-dimensional sections of the body G; W_m–the mean value of the volumes of its m-dimensional projections; w_m–the least of these volumes. In particular, $V_n = W_n = w_n$ is the volume of G, V_1 is the diameter, W_1 is the mean width. (Some inequalities between these quantities and the relation of W_m to the integral curvatures can be found in ([4],[5]). In particular

$$W_{n-1} = \frac{\tau_{n-1}}{\varkappa_n} S,$$

where S is the area of ∂G, $\tau_m^{-1/m} W_m^{1/m} \geq \tau_{m+1}^{-1/(m+1)} W_{m+1}^{1/(m+1)}$.) We shall denote by α_m, etc., positive numbers depending only on m and the dimension n.

(3.1) For every k, obviously, $H_k < V_1$. On the other hand, for $k < 1$, $H_k > \alpha_{kV} 1$, but for $k \geq 1$ such a lower estimate is impossible. (It suffices to note that G contains a segment of length V_1, and therefore, if H'_k is $\max h_k$ for such a segment, then $H_k > H'_k$. Computing H'_k, we obtain what was asserted.)

(3.2) For $k > -1$, $H_k < \frac{1}{2} W_1$, except in the case when G is a ball. By definition,

$$W_1 = \frac{1}{\varkappa_n} \int (h(x,\nu) + h(x,-\nu))\, d\nu = \frac{2}{\varkappa_n} \int h(x,\nu)\, d\nu = 2h_{-1}(x).$$

Therefore $h_{-1}(x) = \frac{1}{2} W_1$, and the assertion follows from (2.2). On the other hand, it also follows from (2.2) that for $k < -1$, $h_k(x) > h_{-1}(x)$, again except in the case of a ball. Therefore, excluding this case, for $k < -1$

$$\min h_k(x) > \frac{1}{2} W_1$$

(cf. this with (2.3)).

(3.3)

$$\alpha_n V_n^{1/n} \geq H_n \geq \beta_n V_n^{1/n}.$$

By virtue of (4) $H_n = \tau_n^{1/n} \widetilde{V}_n^{-1/n}$, where $\widetilde{V}_n = \min_x \widetilde{V}_n(x)$ is the least of the volumes of the polar bodies G^x. Therefore (3.3) is equivalent to

$$\lambda_n \leq V_n \widetilde{V}_n \leq \mu_n, \qquad \lambda_n = \tau_n \alpha_n^{-n}, \quad \mu_n = \tau_n \beta_n^{-n}. \tag{5}$$

The fact that such inequalities for $V_n \widetilde{V}_n$ hold is known and is proved simply. From the known properties of mutually polar bodies there follows the affine invariance of the product $V_n \widetilde{V}_n$. Therefore it is enough to consider bodies of volume $V_n = 1$ contained in a given cube. Then it is obvious that the product $V_n \widetilde{V}_n$ attains finite and positive maximum and minimum values, and this is precisely (5). It is not difficult also to obtain some values for λ_n, μ_n, but the question of the best values remains open (see, for example, ([6]), where $\lambda_n = 4^n (n!)^{-2}$ is given). It is probable that the greatest value of $V_n \widetilde{V}_n$ is attained for an ellipsoid, and the least for a simplex, to which, accordingly, the best values of μ_n, λ_n correspond.

(3.4) If $1 \leq k < n$ and l is the integer part of k, then

$$\alpha'_k W_l^{1/l} \geq H_k \geq \beta'_k W_{l+1}^{1/(l+1)},$$

$$\alpha''_k V_l^{1/l} \geq H_k \geq \beta''_k V_{l+1}^{1/(l+1)} \quad \text{and} \quad H_k \geq \gamma_k w_l,$$

but lower estimates in terms of W_l, V_l are impossible also for $k = l$.

Let us prove the first inequality: $H_k \leq \alpha_k W_l^{1/l}$. Since for $k \geq l$, $H_k \leq H_l$, one may put $k = l$. Let E be an k-dimensional plane, pro-

passing through the center of the sphere Ω; x_E is the projection of the point x onto E; G_E is the projection of G. For $v \in \Omega \cap E$, obviously, $h(x, v) = h(x_E, v)$, that is, the distance from x_E to the supporting plane of G_E with normal v. Therefore, if we put

$$h_k(x, E) = \left[\frac{1}{\chi_k} \int_{\Omega \cap E} h^{-k}(x, v)\, dv\right]^{-1/k}, \tag{6}$$

then $h_k(x, E)$ will be nothing other than the function h_k for G_E. Hence, according to (3.3), we conclude that $h_k(x, E) = h_k(x_E, E) \leq \alpha_k V_k^{1/k}(E)$, where $V_k(E)$ is the volume of G_E.

If the measure in the set $\mathcal{E}$ of planes E is normalized so that $\operatorname{mes} \mathcal{E} = 1$, then for every integrable $f(v)$ we have

$$\frac{1}{\chi_n} \int_\Omega f(v)\, dv = \int_{\mathcal{E}} \left(\frac{1}{\chi_k} \int_{\Omega \cap E} f(v)\, dv\right) dE.$$

Applying this equality to $f(v) = h^{-k}(x, v)$ and using (6), we obtain

$$h_k^{-k}(x) = \frac{1}{\chi_n} \int_\Omega h^{-k}(x, v)\, dv = \int_{\mathcal{E}} \left(\frac{1}{\chi_k} \int_{\Omega \cap E} h^{-k}(x, v)\, dv\right) dE = \int_{\mathcal{E}} h_k^{-k}(x, E)\, dE,$$

and since $h_k(x, E) \leq \alpha_k V_k^{1/k}$, i.e. $h_k^{-k}(x, E) \geq \alpha_k^{-k} V_k^{-1}(E)$, it follows that

$$h_k^{-k}(x) \geq \alpha_k^{-k} \int_{\mathcal{E}} V_k^{-1}(E)\, dE \geq \alpha_k^{-k} \left(\int_{\mathcal{E}} V_k(E)\, dE\right)^{-1} = \alpha_k^{-k} W_k^{-1},$$

where we have used the well-known inequality for means and the definition of the quantity W_k. Thus, for every $x \in G$ we have $h_k(x) \leq \alpha_k W_k^{1/k}$, as was required to prove.

We omit the proofs of the other inequalities (3.4). The inequalities with W and V follow from one another, since $\delta_m \geq W_m V_m^{-1} \geq \delta'_m$; moreover, $W_{m+1} \geq \delta''_m w_m$.

4. We now indicate functions estimating $h_k(x)$, $0 < k \leq n$.

(4.1) Let G_m be the body symmetric to G with respect to the point $x \in G$, and let $V(x)$ be the volume of the body $H = G \cap G_x$. It turns out that for $h_n(x)$

$$\bar{\alpha}_n V_n^{1/n}(x) \geq h_n(x) \geq \beta_n V^{1/n}(x), \tag{7}$$

where, incidentally, β_n is the same as in (3.3). (From the left inequality (7) it is not difficult to conclude that if $\bar{V}(x)$ is the smallest of the volumes cut off from G by arbitrary planes passing through x, then $h_n(x) \le \alpha_n 2^{1/n}\bar{V}^{1/n}(x)$.)

Let us prove (7). Let $\bar{h}_n$ be the function h_n for the body H. Obviously, $h_n(x) \ge \bar{h}_n(x)$, and by (3.3) $\bar{h}_n(x) \ge \beta_n V^{1/n}(x)$, whence the right-hand inequality (7) follows.

For the proof of the left-hand inequality we use the lemma:

Lemma. If K is a convex body, K_x is the body symmetric to it with respect to the point $x \in K$, and $\bar{K}$ is the convex hull of $K \cup K_x$, then

$$V(K) \ge \gamma_n V(\bar{K}). \tag{8}$$

For the proof, observe that, by the affine invariance of the ratio of volumes, one may assume $V(K) = 1$ and that K is contained in some ball. On the set of bodies K satisfying these conditions, for all $x \in K$, the volumes $V(\bar{K})$ are bounded. If γ_n^{-1} is their least upper bound, then (8) holds.

We apply this lemma to $K = \widetilde{G}$, where $\widetilde{G}$ is the body polar to G with respect to the point x. Then $K_x = \widetilde{G}_x$ is the body polar to G_x. From the properties of polarity we conclude that $\bar{K}$ is the body $\widetilde{H}$, polar to $H = G \cap G_x$ (because under polarity the common points of bodies correspond to planes not intersecting their polars). Therefore (8) gives:

$$V(\widetilde{G}) \ge \gamma_n V(\widetilde{H}). \tag{9}$$

But, as a consequence of (4), $\bar{V}(\bar{G}) = \tau_n \bar{h}_n^{-n}(x)$, $\bar{V}(\bar{H}) = \tau_n \bar{h}_m^{-n}(x)$. Therefore, from (9) it follows that $\bar{h}_n(x) \ge \gamma_n^{1/n} h_n(x)$, and since, by (3.3), $\bar{h}_n(x) \le \bar{a}_n V^{1/n}(x)$, we obtain the left-hand inequality (7). (The best values of the constants in (7), (8) are unknown. The best γ_n is the minimum of $V(\bar{K}) : V(K)$; it is probably attained when K is a simplex and x is one of its vertices.)

(4.2) Let $h(x)$ be the distance from $x \in G$ to ∂G, i.e.

$$h(x) = \min_v h(x, v).$$

For every $k > 0$ and $\le n$,

$$h_k(x) \le h^{1-n/k}(x) h_n^{n/k}(x) \le \bar{a}_n^{n/k} h^{1-n/k}(x) V^{1/k}(x), \tag{10}$$

where $V(x)$, $\bar{a}_n$ are the same as in (4.1).

The second inequality is a consequence of (7). The first is obtained as follows: in view of (1) and the fact that $h(x) \le h(x, v)$,

$$h_k(x) = h^{1-n/k}(x) \left[\frac{1}{\chi_n} \int_\Omega h^{k-n}(x) h^{-k}(x, v)\, dv \right]^{-1/k} \le h^{1-n/k}(x) h_n^{n/k}(x).$$

Analogously one obtains the lower estimate with $H(x) = \max h(x, v)$:

$$h_k(x) \geq H^{1-n/k}(x) h_n^{n/k}(x) \geq \beta_n^{n/k} H^{1-n/k}(x) V^{1/k}(x).$$

Estimate (10), in particular, determines when and with what rate $h_k(x) \to 0$, if $x \to x_0 \in \partial G$. If at x_0 the body G is, so to speak, sufficiently convex, then $V(x) \to 0$ so rapidly that also $h^{1-n/k}(x) V^{1/k}(x) \to 0$. For example, from (10) it is easy to conclude that for $k > n - 1$

$$h_k(x) \leq A h^{1-(n-1)/k}(x), \qquad A = A(n, k, V_1).$$

If, however, $k = n - 1$, then $h_k(x)$ vanishes at all points of ∂G, except those lying inside flat faces.

Further, if G is contained in a paraboloid of degree l with vertex $x_0 \in \partial G$, then

$$h_k(x) \leq A h^q(x), \qquad q = 1 - (n-1)(l-1)/kl.$$

If G contains a similar paraboloid with vertex x_0, then for points on its axis near x_0,

$$h_k(x) \leq A_1 h^q(x).$$

This shows that estimate (10) in this case gives the correct order of magnitude of $h_k(x)$.

(4.3) If the point $x_0 \in \partial G$ is conical, then

$$h_0(x) \leq A |x - x_0|^{\omega/\chi_n},$$

where ω is the solid angle of the normals to the supporting planes at x_0. (The proof consists in estimating $h_0(x)$ for a cone.) If, however, $x_0 \in \partial G$ is the vertex of a paraboloid of degree $l > 1$ contained in G, then $h_0(x_0) > 0$. For $k < 0$, $h_k(x) > 0$ everywhere on ∂G. Let us also note that on the surface of the unit sphere

$$\ln h_0(x) = (-1)^{n-1} \left(\ln 2 + \sum_{m=1}^{n-2} \frac{(-1)^m}{m} \right).$$

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