# ON THE DENSITY HYPOTHESIS OF E. BOMBIERI

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## Abstract

## Full Text

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**MATHEMATICS**

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# ON THE DENSITY HYPOTHESIS OF E. BOMBIERI

*(Presented by Academician Yu. V. Linnik on 9 IV 1966)*

In a recent work ([1]), Bombieri, generalizing Yu. V. Linnik' s "large sieve" method ([2]), proved an elegant density theorem, a simple consequence of which is the following form of the "averaged" law of distribution of prime numbers in arithmetic progressions:

$$\sum_{D\le x^{1/2}/(\ln x)^B} \max_{(l,D)=1}\left|\pi(x,D,L)-\frac{\operatorname{li} x}{\varphi(D)}\right| = O\left(\frac{x}{\ln^A x}\right) \qquad (1)$$

for any arbitrarily large but fixed $A$, and for some $B=B(A)$. Estimate (1) replaces the extended Riemann hypothesis in well-known additive problems ([3],[4]). (An estimate close to (1) was obtained by A. I. Vinogradov in ([5]), by another method and independently; there an interesting application to the generalized divisor problem of Hooley–Linnik is also given.)

Many problems of this type were first solved by Yu. V. Linnik' s dispersion method ([6]), which in essence uses the achievements of modern algebraic geometry and whose natural sphere of action is formed by binary problems not derivable from the extended Riemann hypothesis, for example,

$$N = Q(x,y)+P(p_1,p_2),$$

where $Q(x,y)$, $P(x,y)$ are positive definite quadratic forms; $p_1,p_2$ are prime numbers, $p_1\le N^{\alpha}$, $p_2\le N^{1-\alpha}$.

In a somewhat roughened form Bombieri' s density theorem has the form

$$\sum_{D\le X}\sum_{\chi'_D} N(a,T,\chi'_D) \ll X^{8(1-\alpha)/(3-2\alpha)+\varepsilon}T^c, \qquad (2)$$

where $N(a,T,\chi)$ denotes the number of zeros of the Dirichlet $L$-series $L(s,\chi)$ in the rectangle $a\le\sigma\le 1$, $|t|\le T$, with $s=\sigma+it$, $T\ge 2$, $a\ge 1/2$; $\varepsilon$ is arbitrarily small but fixed; $\chi'_D$ is a primitive character mod$D$.

In the same work ([1]), Bombieri formulates the hypothesis

$$\sum_{D\le X}\sum_{\chi'_D} N(a,T,\chi'_D) \ll X^{4(1-\alpha)+\varepsilon}T^{1+\varepsilon}. \tag{3}$$

Results of type (2), (3) obviously make it possible to conclude that, for almost all moduli $D$, the quasi-Riemann hypothesis holds (for all $\chi'_D$, $L(s,\chi'_D)$ have no zeros in the rectangle $\sigma > \gamma_0,\ |t| \le D^{\varepsilon}$, $\gamma_0 < 1$ an absolute constant). This, in turn, makes it possible to obtain a power saving in the asymptotic formula for the moments of the number of ideal classes of a quadratic field.

The following two inequalities constitute a strengthening of Bombieri' s estimate (2), useful for the applications described.

**Theorem 1.** *The inequalities hold*

$$\sum_{D\le X}\sum_{\chi'_D} N(a,T,\chi'_D) \ll X^{10(1-\alpha)/(3-\alpha)+\varepsilon}T^{c}, \tag{4}$$

$$\sum_{D\le X}\sum_{\chi'_D} N(\alpha,T,\chi'_D) \ll X^{10/4(1-\alpha)+\varepsilon}T^{c} \tag{5}$$

uniformly for all $\alpha,\ 1/2 \le \alpha \le 1$.

Inequality (4) is obtained by Gabriel's method (see (7), p. 238) and the following estimate of the eighth moment of $L$-series on the critical line:

$$\sum_{D\le X}\sum_{\chi_D\ne\chi_0} |L(1/2+it,\chi)|^8 \ll X^{2+\varepsilon}(|t|+2)^{c} \tag{6}$$

(an obvious consequence of Theorem 3 of Bombieri's paper (1) and the truncated functional equation for Dirichlet $L$-series).

Inequality (5) is proved according to the classical scheme (see (8)). Here the main role is played by Burgess' s estimate (9) for the modulus of the $L$-series on the line $\sigma = 1/2$.

An obvious consequence of inequality (4) of Theorem 1 is

**Theorem 2.** For all moduli $D$ in the interval $[1, X]$, with the exception of at most $X^{1-\delta}$, all $L$-series corresponding to primitive characters mod$D$ have no zeros in the region $|\sigma| > 7/9+\delta,\ |t| \le D^{\gamma}$ $(\varepsilon = \varepsilon(\delta,\gamma))$.

Using Theorem 2 according to the scheme of paper (10), we obtain the following theorem.

**Theorem 3.** Let $h(-D)$ denote the number of classes of purely radical quadratic forms of determinant $-D,\ D > 0$. Then for any fixed $k$ one has

$$\sum_{D\le N} h^k(-D) = \frac{2^{k+1}r(k)}{\pi^k(k+2)} N^{(k+2)/2}\{1+O(N^{-\xi})\}, \tag{7}$$

where

$$r(k) = \sum_{\substack{n=1\\ n\equiv 1\ (2)}}^{\infty} \frac{\varphi(n)\tau_k(n^2)}{n^3}, \qquad \xi \text{ is any constant smaller than}$$

$$(\sqrt{129}-9)/2 \approx 0.18.$$

A substantial advance in this question would now be a proof of (7) with $\xi = 1/2+\varepsilon$ (Bombieri' s hypothesis (3) gives $\xi = 1/5+\varepsilon$).

In conclusion we formulate the hypothesis.

**Hypothesis.** The inequality

$$\sum_{D\le X} \max_{\chi'_D} N(\alpha, T, \chi'_D) \ll X^{2(1-\alpha)+\varepsilon} T^c$$

holds uniformly for $1/2 \le \alpha \le 1$.

Let us note that a consequence of this hypothesis is the absence (for almost all moduli $D$) of zeros of the functions $L(s,\chi'_D)$ in the rectangle $\sigma > 1/2+\varepsilon$; $|t| \le D^\gamma$, as well as the possibility of choosing $\xi = 1/3+\varepsilon$ in Theorem 3.

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*Note: Figure translations are in progress. See original paper for figures.*