# Mathematics

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**Abstract**

**Full Text**

*Mathematics*

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# ON A BOUNDARY-VALUE PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION OF ORDER $n$

*(Presented by Academician L. S. Pontryagin, 8 XII 1966)*

Consider the boundary-value problem

$$x^{(n)} = f(t, x, \dots, x^{(n-1)}), \tag{1}$$

$$x^{(\nu)}(0) = a_\nu, \qquad x^{(k)}(\tau) = b, \qquad \nu = 0, 1, \dots, n-2, \quad 0 \le k \le n-3, \tag{2}$$

under the assumption that the function $f(t, x_0, \dots, x_{n-1})$ is defined in the domain $D_\tau(0 \le t \le \tau,\ -\infty < x_0, \dots, x_{n-1} < \infty)$, is measurable in $t$ for fixed $x_0, \dots, x_{n-1}$, continuous in $x_0, \dots, x_{n-1}$ for fixed $t$, and for any bounded domain $G$ in the space $x_0, \dots, x_{n-1}$ there exists a summable function $g(t) \ge 0$ such that $|f(t, x_0, \dots, x_{n-1})| \le g(t)$, $0 \le t \le \tau$, $(x_0, \dots, x_{n-1}) \in G$ (Carathéodory conditions). In addition, in proving certain assertions we shall assume that the function $f$ satisfies a generalized local Lipschitz condition in the variables $x_0, \dots, x_{n-1}$, i.e., for each bounded domain $G$ in the space $x_0, \dots, x_{n-1}$ there exists a summable function $L(t) \ge 0$ such that

$$|f(t, x'_0, \dots, x'_{n-1}) - f(t, x''_0, \dots, x''_{n-1})| \le L(t) \sum_{k=0}^{n-1} |x'_k - x''_k|,$$

$$0 \le t \le \tau, \qquad (x'_0, \dots, x'_{n-1}) \in G, \quad (x''_0, \dots, x''_{n-1}) \in G.$$

**Theorem 1.** Suppose the function $f(t, x_0, \dots, x_{n-1})$ satisfies the Carathéodory conditions and the following conditions:

$(A_n)$. There exists $h > 0$ such that

$$f(t, x_0, \dots, x_{n-2}, 0)x_{n-2} \ge 0 \quad \text{for } |x_{n-2}| \ge h;$$

$$0 \le t \le \tau, \qquad -\infty < x_0, \dots, x_{n-3} < \infty.$$

$(B_n^k)$. For every $R > 0$ there exist summable functions $a_0(t) \ge 0$, $a_1(t) \ge 0$, $b_r(t) \ge 0$, $0 \le t \le \tau$, $r = 1, \dots, m$, a continuous function $b(z) \ge 0$, $0 \le z < \infty$, and numbers $0 < \alpha_1 < \alpha_2 < \dots < \alpha_m < 1/(n-k-1)$ such that:

1) $b(z) \to 0$ as $z \to \infty$;

2) for any $t_0 \in [0, \tau]$

$$\left| \int_t^{t_0} b_r(s)\, ds \right| \le \beta_r(|t_0 - t|)\, |t_0 - t|^{(n-k-1)\alpha_r}, \qquad t \in [0, \tau],$$

where $\beta_r(t) \ge 0$, $r = 1, \dots, m$, $0 \le t \le \tau$, are continuous monotonically decreasing functions satisfying the condition $\beta_r(t) \to 0$ as $t \to 0$;

3) when $|x_0| + \dots + |x_k| \le R,\ 0 \le t \le \tau$,

$$|f(t, x_0, \dots, x_{n-1})| \le a_0(t) + a_1(t) z + \sum_{r=1}^{m} b_r(t) z^{1+\alpha_r} + b(z) z^{(n-k)/(n-k-1)},$$

where

$$z = |x_{k+1}|^{\,n-k-1} + |x_{k+2}|^{(n-k-1)/2} + \dots + |x_{k-1}|.$$

Then a solution of the problem (1), (2) exists.

The proof of the theorem is based on paper ([1]) and on the following lemmas.

**Lemma 1.** *Let $x(t)$, $0 \le t \le \tau$, be a solution of problem (1), (2), where the function $f(t, x_0, \dots, x_{n-1})$ satisfies condition $(A_n)$ and the generalized local Lipschitz condition with respect to $x_{n-1}$.*

*Then there exists a constant $M > 0$, depending on $h$ and on the constants $a_\nu$, $\nu = 0, 1, \dots, n-2$, $b$, $\tau$, such that*

$$|x(t)| \le M, \qquad 0 \le t \le \tau.$$

We note that Lemma 1 ceases to be valid if the function $f$ does not satisfy the generalized local Lipschitz condition with respect to $x_{n-1}$.

**Lemma 2.** *Let $x(t)$, $0 \le \tau_1 \le t \le \tau_2 \le \tau$, be a solution of equation (1), where the function $f$ satisfies condition $(B_n^k)$. Then for every $M > 0$ one can indicate an $N > 0$ such that if $|x^{(k)}(t)| < M$, then $|x^{(n-1)}(t)| < N$, $\tau_1 \le t \le \tau_2$.*

We note that condition $(B_n^k)$ is sharp and in the general case cannot be improved. As functions $b_r(t)$ satisfying condition $(B_n^k)$, one may take, for example, functions of the form

$$b_r(t) = \sum_{s=1}^{p} \frac{b_{r,s}(t)}{|t - t_{r,s}|^{1-(n-k-1)\alpha_r}}, \qquad t_{r,s} \in [0, \tau],$$

where $b_{r,s}(t) \geq 0$, $s = 1, \dots, p$, $0 \leq t \leq \tau$, are continuous functions, and $b_{r,s}(t_{r,s}) = 0$. Lemma 2 strengthens the corresponding results of papers ([2],[3] ).

**Theorem 2.** *If the function $f(t, x_0, \dots, x_{n-1})$ is nondecreasing in $x_0, \dots, x_{n-2}$ and satisfies the generalized local Lipschitz condition in $x_0, \dots, x_{n-1}$, then problem (1), (2) cannot have two solutions.*

**Remark.** The assertion of Theorem 1 remains valid if conditions (2) are replaced by the somewhat more general ones:

$$x^{(\nu)}(0) = \varphi_\nu(x^{(n-1)}(0)), \qquad x^{(k)}(\tau) = -\varphi(x^{(n-1)}(\tau)) \tag{3}$$

$$\nu = 0, 1, \dots, n-2; \quad 0 \leq k \leq n-3,$$

where $\varphi_\nu(s)$ and $\varphi(s)$, $-\infty < s < \infty$, are continuous functions, bounded above for $s \leq 0$ and bounded below for $s \geq 0$. If $\varphi_\nu(s)$ and $\varphi(s)$ are nondecreasing functions of $s$ and the conditions of Theorem 2 are fulfilled, then problem (1), (3) cannot have two solutions.

**Theorem 3.** *Consider the boundary conditions*

$$x^{(\nu)}(0) = a_\nu, \qquad x^{(n-2)}(\tau) = b, \qquad \nu = 0, \dots, n-2. \tag{4}$$

*A solution of problem (1), (4) exists if the function $f(t, x_0, \dots, x_{n-1})$ satisfies condition $(A_n)$ of Theorem 1 and condition $(B_n^{n-2})$. For every $R > 0$ there exist summable functions $a_0(t) \geq 0$, $a_1(t) \geq 0$, $b_r(t) \geq 0$, $r = 1, \dots, m$, $0 \leq t \leq \tau$, a continuous positive nondecreasing function $b(z) > 0$, $0 \leq z < \infty$, and constants $a_k > 0$, $0 < \alpha_1 < \dots < \alpha_n = 1$, such that*

$$|f(t, x_0, \dots, x_{n-1})| \leq a_0(t) + a_1(t)|x_{n-1}| + \sum_{r=1}^{m} b_r(t)|x_{n-1}|^{1+\alpha_r} +$$

$$+ b(|x_{n-1}|)x_{n-1}^2,$$

$$0 \leq t \leq \tau, \qquad |x_0| + \dots + |x_{n-2}| < R, \qquad -\infty < x_{n-1} < \infty,$$

*and, moreover, for every $t_0 \in [0, \tau]$*

$$\int_t^{t_0} \frac{ds}{\left|\sum_{r=1}^{m} \int_s^{t_0} b_r(u)\, du\right|^{1/\alpha_r} + \left|\int_s^{t_0} b(|\dot{v}(u)|)\, du\right|} = \infty,$$

*where* $\dot{v}(t)$ *is a solution of the equation*

$$\ddot{v} = b(|\dot{v}|)\dot{v}^2,$$

*satisfying the condition*

$$\dot{v}(t) \to \infty \quad \text{as } t \to t_0.$$

We note that condition ($\mathrm{B}_n^{n-2}$) is also sharp and in the general case cannot be improved. As functions $b_r(t)$, $0 \le t \le \tau$, satisfying condition ($\mathrm{B}_n^{n-2}$), one may take, for example, functions of the form

$$b_r(t) = \sum_{s=1}^{p} \frac{b_{r,s}(t)}{|t - t_{r,s}|^{1-\alpha_r}} \, |\ln|t - t_{r,s}||^{\beta_r}, \quad t_{r,s} \in [0, \tau],$$

where $b_{r,s}(t)$ are continuous functions and $\beta_r \le \alpha_r$. In this case, as the function $b(z)$ one may take, for example, the function $b(z) = c_1 + c_2 \ln(1 + |z|)$, where $c_1 \ge 0$ and $c_2 \ge 0$ are arbitrary constants.

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*Note: Figure translations are in progress. See original paper for figures.*