INDUCTIVE AND PROJECTIVE LIMITS OF COMMUTATIVE TOPOLOGICAL GROUPS

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Abstract

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INDUCTIVE AND PROJECTIVE LIMITS OF COMMUTATIVE TOPOLOGICAL GROUPS

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We shall consider three categories of commutative groups in which there exist limits of direct and inverse spectra, as well as sums and products. A topology t on a group X will be called a **group topology** if (X,t) is a topological group. R denotes the additive group of real numbers; Z the subgroup of integers; $T = R/Z$; $\vartheta : R \to T$ the canonical representation, and $W = \vartheta([-1/4, 1/4])$.

1. Let X' be the group of all characters of the topological group* (X,t) (i.e. of its continuous representations in T); $\langle x, x' \rangle$ is the value of the character $x' \in X'$ at the point $x \in X$; $M^0 = \{x' \in X' : \langle M, x' \rangle \subset W\}$ for any set $M \subset X$, and ${M'}^0 = \{x \in X : \langle x, M' \rangle \subset W\}$ for any set $M' \subset X'$. A set $M \subset X$ is called, following N. Ya. Vilenkin ([1]), **quasi-convex** if $(M^0)^0 = M$. If φ is a continuous representation of (X,t) in (Y,v), then $\varphi(M^{00}) \subset (\varphi M)^{00}$ for any set $M \subset X$, and $\varphi^{-1}N$ is quasi-convex for any quasi-convex set $N \subset Y$. The group (X,t) is called **locally quasi-convex** (an lc-group) ([1]) if in (X,t) there exists a fundamental system of neighborhoods of zero consisting of quasi-convex sets. The topology of an lc-group is called an lc-topology. If (X,t) is a separated lc-group, then for every $x \in X \setminus \{0\}$ there exists an $x' \in X'$ such that $\langle x, x' \rangle \neq 0$.

Theorem 1. Let $(X_\alpha, t_\alpha)_{\alpha \in A}$ be an arbitrary family of topological groups, X a group, and for every $\alpha \in A$ let a representation $\varphi_\alpha : X_\alpha \to X$ be defined.

Then among those group (respectively lc-) topologies on X for which all φ_α are continuous, there exists a strongest topology t (respectively an lc-topology c), and

$$(X,c)' = (X,t)'.$$

If $\mathfrak{B}$ is a fundamental system of neighborhoods of zero in (X,t), then $(U^{00})_{U \in \mathfrak{B}}$ is a fundamental system of neighborhoods of zero in (X,c). If (Y,v) is an (lc-)group, then the representation $\varphi : (X,t) \to (Y,v)$ $((X,c) \to (Y,v))$ is continuous if and only if $\varphi\varphi_\alpha$ is continuous for every $\alpha \in A$.

Theorem 2. Let $(X^\alpha, t^\alpha)_{\alpha \in A}$ be an arbitrary family of lc-groups, X a group, and for every $\alpha \in A$ let a representation $\varphi^\alpha : X \to X^\alpha$ be defined.

Then the weakest topology t on X for which all φ^α are continuous is an lc-topology.

A **boundedness** in a group X ([1]) is a set $\mathfrak{B}X$ of subsets of X satisfying the following axioms: $M \cup N \in \mathfrak{B}X$ and $M + N \in \mathfrak{B}X$ for any $M, N \in \mathfrak{B}X$; $\bigcup_x M = X$; if $M \subset N$ and $N \in \mathfrak{B}X$, then $M \in \mathfrak{B}X$; if $M \in \mathfrak{B}X$, then $-M \in \mathfrak{B}X$. A set $\mathfrak{B}_1 X \subset \mathfrak{B}X$ is called a **base of the boundedness** $\mathfrak{B}X$ if for every $M \in \mathfrak{B}X$ there exists an $N \in \mathfrak{B}_1 X$ such that $M \subset N$. Examples of boundednesses are the sets $b(X, t)$ of all bounded sets ([2]) and $pc(X, t)$ of all precompact sets of the topological group (X, t); the set $c(X, t)$ of all bicompact sets is a base of boundedness

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* Here, as everywhere below, by a "group" is meant a commutative group.

in X. The group X, endowed with the boundedness $\mathfrak{B}X$, will be denoted by $[X, \mathfrak{B}X]$. A representation

$$\varphi : [X, \mathfrak{B}X] \to [Y, \mathfrak{B}Y]$$

is called **bounded** if $\varphi M \in \mathfrak{B}Y$ for every $M \in \mathfrak{B}X$. The groups $[X, \mathfrak{B}X]$ and $[Y, \mathfrak{B}Y]$ are called **isomorphic** if there exists such an algebraic isomorphism φ of X onto Y that φ and φ^{-1} are bounded representations. On the set of all boundednesses of the group X an order relation is introduced by inclusion. A boundedness $\mathfrak{B}X$ in a topological group (X, t) is called **quasi-convex** if for $\mathfrak{B}X$ there exists a base consisting of quasi-convex sets. If $\mathfrak{B}X$ is a boundedness in (X, t), then the family $(M^{00})_{M \in \mathfrak{B}X}$ is a base of a quasi-convex boundedness in X. We shall denote it by $\mathfrak{B}^{00}X$. An lc-group endowed with a quasi-convex boundedness is called an lcb-group.

Theorem 3. Let $[X_\alpha, t_\alpha, \mathfrak{B}X_\alpha]_{\alpha \in A}$ be a family of lcb-groups; let X be a group; for every $\alpha \in A$ a representation $\varphi_\alpha : X_\alpha \to X$ is defined, and let t be the strongest lc-topology in X for which all φ_α are continuous.

Then in X there exists the weakest boundedness $\mathfrak{B}X$ for which all φ_α are bounded representations, and $(M^{00})_{M \in \mathfrak{B}X}$ is a base of the weakest quasi-convex boundedness in X for which all φ_α are bounded representations. If $[Y, v, \mathfrak{B}Y]$ is an lcb-group, then

$$\varphi : X \to Y$$

is a continuous bounded representation if and only if $\varphi\varphi_\alpha$ is a continuous bounded representation for every $\alpha \in A$.

Theorem 4. Let $[X^\alpha, t^\alpha, \mathfrak{B}X^\alpha]_{\alpha \in A}$ be a family of lcb-groups; let X be a group; for every $\alpha \in A$ a representation

$$\varphi^\alpha : X \to X^\alpha$$

is defined, and let t be the weakest lc-topology in X for which all φ_α are continuous.

Then in X there exists the strongest boundedness $\mathfrak{B}X$ for which all φ^α are bounded representations, and moreover $\mathfrak{B}X$ is a quasi-convex boundedness. If $[Y, v, \mathfrak{B}Y]$ is an *lcb*-group, then

$$\varphi : Y \to X$$

is a continuous bounded representation if and only if $\varphi^\alpha \varphi$ is a continuous bounded representation for every $\alpha \in A$.

Let $\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{K}_3$ be, respectively, the categories of all (commutative) topological groups, *lc*-groups, and *lcb*-groups, whose morphisms are all possible continuous representations in the case of $\mathcal{K}_1$ and $\mathcal{K}_2$, or continuous bounded representations in the case of $\mathcal{K}_3$. In each of these categories there exist limits of direct spectra and sums (realized on the basis of Theorem 1 or 3), and also limits of inverse spectra and products (realized on the basis of Theorem 2 or 4). Thus, the limit of the direct spectrum

$$\{[X_\alpha, t_\alpha, \mathfrak{B}X_\alpha]; \pi^\alpha_\beta\}_A$$

in $\mathcal{K}_3$ is its limit X in the category of groups, endowed with the strongest *lc*-topology t and the weakest quasi-convex boundedness $\mathfrak{B}X$, for which all canonical representations

$$\pi_\alpha : X_\alpha \to X$$

are continuous and bounded. The sum of an arbitrary family of *lcb*-groups

$$[X_\alpha, t_\alpha, \mathfrak{B}X_\alpha] \quad (\alpha \in A)$$

in $\mathcal{K}_3$ is the sum

$$\sum_{\alpha \in A} X_\alpha$$

of this family in the category of groups, endowed with the strongest *lc*-topology and the weakest quasi-convex boundedness for which all canonical representations

$$\iota_\alpha : X_\alpha \to \sum_{\alpha \in A} X_\alpha$$

are continuous and bounded. In $\mathcal{K}_3$ the quotient group of an *lcb*-group $[X, t, \mathfrak{B}X]$ by a subgroup Y is defined as the quotient group X/Y, endowed with the strongest *lc*-topology and the weakest quasi-convex boundedness for which the canonical representation

$$\omega : X \to X/Y$$

is continuous and bounded. Similarly, in $\mathcal{K}_3$ the limit of an inverse spectrum and the product, which is determined by a subgroup, are realized. Many results known for limits of direct and inverse spectra in the category of groups ([3]) are also valid in $\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{K}_3$. Moreover, for some classes of spectra ([4,5]) the limits have such important properties as separability, completeness, and others.

2. In this part we consider the category $\mathscr{K}_3$. Let $[X, t, \mathfrak{B}X]$ be an *lcb*-group; its character group X', endowed with the *lc*-topology in which

a fundamental system of neighborhoods of zero consists of sets of the form M^0, where $M \in \mathfrak{B}X$, and a basis of the quasi-convex boundedness consists of sets of the form U^0, where U is a neighborhood of zero in (X, t), is called the group **conjugate** to $[X, t, \mathfrak{B}X]$, and is denoted by $[X, t, \mathfrak{B}X]'$ (1). $[X, t, \mathfrak{B}X]'$ is a separable *lcb*-group. If

$$\varphi : [X, t, \mathfrak{B}X] \to [Y, v, \mathfrak{B}Y]$$

is a continuous bounded representation, then also $\varphi' : Y' \to X'$, defined by the formula

$$\langle x, \varphi' y' \rangle = \langle \varphi x, y' \rangle \quad (x \in X,\ y' \in Y'),$$

is a continuous bounded representation. φ' is called **conjugate** to φ. A separable *lcb*-group $[X, t, \mathfrak{B}X]$ is called **reflexive** if the representation $x \mapsto \langle x, \cdot \rangle$ is an isomorphism of the group $[X, t, \mathfrak{B}X]$ onto $[X, t, \mathfrak{B}X]''$.

Theorem 5. Let $\{[X_\alpha, t_\alpha, \mathfrak{B}X_\alpha]; \pi^\alpha_\beta\}_A$ be a direct spectrum in $\mathfrak{K}_3$. If $X'_\alpha \neq \{0\}$ $(\alpha \in A)$, then

$$(\varinjlim\{[X_\alpha, t_\alpha, \mathfrak{B}X_\alpha]; \pi^\alpha_\beta\}_A)'$$

is isomorphic to

$$\varprojlim\{[X_\alpha, t_\alpha, \mathfrak{B}X_\alpha]'; (\pi^\beta_\alpha)'\}_A.$$

Theorem 6. Let $\{[X^\alpha, t^\alpha, \mathfrak{B}X^\alpha]; \rho^\beta_\alpha\}_A$ be an inverse spectrum in $\mathfrak{K}_3$ and

$$\varprojlim\{[X^\alpha, t^\alpha, \mathfrak{B}X^\alpha]; \rho^\beta_\alpha\}_A = [X, t, \mathfrak{B}X].$$

If the projection $\rho^\alpha X$ is dense in (X^α, t^α) $(\alpha \in A)$, then there exists a continuous $\mathfrak{B}$-isomorphism of the group

$$\varinjlim\{[X^\alpha, t^\alpha, \mathfrak{B}X^\alpha)'; (\rho^\beta_\alpha)'\}_A$$

onto $[X, t, \mathfrak{B}X]'$. If, moreover, all $[X^\alpha, t^\alpha, \mathfrak{B}X^\alpha]$ are reflexive, $\mathfrak{B}X^\alpha \subset pc^{00}(X^\alpha, t^\alpha)$ and $\mathfrak{B}X \supset c(X, t)$, then $[X, t, \mathfrak{B}X]'$ is isomorphic to

$$\varinjlim\{[X^\alpha, t^\alpha, \mathfrak{B}X^\alpha]'; (\rho^\beta_\alpha)'\}_A.$$

3. A group of the I (II) kind (6) will mean a topological group isomorphic to the limit of a direct* (inverse) countable spectrum of separable locally bicompact groups (the formation of these limits in the categories of topological spaces $\mathfrak{K}_1$ or $\mathfrak{K}_2$ leads to isomorphic topological groups; in the case $\mathfrak{K}_3$ every topological group (X, t) is endowed with the quasi-convex boundedness $c^{00}(X, t)$). The sum (product) of a countable family of real lines is not a locally bicompact group of the I (II) kind. This shows that there

exist groups of the I (II) kind which are not groups of the II (I) kind. Denote by A and B the classes of all groups of the I and, respectively, II kind. $A' = B$, $B' = A$, and every group of these classes is c^{00}-reflexive. The classes A and B are closed with respect to passage to closed subgroups and to quotient groups by them. If X is a group of the I or II kind and Y is its closed subgroup, then

$$(X/Y)' \cong Y^0$$

and

$$Y' \cong X'/Y^0$$

(an isomorphism in $\mathfrak{K}_3$). For any $y' \in Y'$ and $x \in X \setminus Y$ there exists an $x' \in X'$ such that $x'|_Y = y'$ and $\langle x, x' \rangle \neq 0$. The last result (for groups of the I kind) is also contained in (7), but is proved there by another method (we used the fact that a quotient group of a group of the II kind is a group of the II kind).

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References

1. N. Ya. Vilenkin, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **15**, 5, 439 (1951).
2. J. Hejcman, *Czechoslovak Math. J.*, **9** (84), 4, 544 (1959).
3. N. Steenrod, S. Eilenberg, *Foundations of Algebraic Topology*, Moscow, 1958.
4. I. V. Karklin' sh, *Latvian Math. Collection*, **5**, 1, 57 (1965).
5. I. V. Karklin' sh, *Latvian Math. Yearbook*, **347** (1966).
6. S. Kaplan, *Duke Math. J.*, **17**, 4, 419 (1950).
7. N. Th. Varopoulos, *Proc. Cambridge Phil. Soc.*, **60**, 3, 465 (1964).

* The direct spectrum $\{X_\alpha; \pi_\beta^\alpha\}$ is considered here with monomorphisms π_β^α.

Note: Figure translations are in progress. See original paper for figures.